WEEK 9

SUMMARY OF PARTICLE IN CENTRAL POTENTIAL

$V(\vec{x}) = V(r) \Rightarrow [\hat{H}, \hat{\vec{L}}^2] = [\hat{H}, \hat{L}_z] = [\hat{\vec{L}}^2, \hat{L}_z] = 0$

SIMULTANEOUS EIGEN STATES OF $\{\hat{H}, \hat{\vec{L}}^2, \hat{L}_z\}$

$$\psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)\, Y_{lm}(\theta,\varphi)$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\hat{H}\psi_{nlm} = E_{nl}\psi_{nlm}$$

$$\hat{H} = -\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\hat{\vec{L}}^2}{2m_e r^2} + V(r)$$

SEPARATION OF VARIABLES $\psi_{nlm} = R_{nl} Y_{lm}$

RADIAL EQN.

$$-\frac{\hbar^2}{2m_e}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)R_{nl} + \left(\frac{\hbar^2 l(l+1)}{2m_e r^2} + V(r)\right)R_{nl} = E_{nl}R_{nl}$$

does not depend on m! $\hat{L}_z$ quantum nr.

m_e ... mass of particle (electron)

ANGULAR EQN.

$$\hat{\vec{L}}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1)\, Y_{lm}(\theta,\varphi)$$

IMPORTANT
- R_{nl}, E_{nl} DEPEND ON CHOICE OF POTENTIAL $V(r)$
- Y_{lm} ARE UNIVERSAL DO NOT DEPEND ON $V(r)$

EX: HYDROGEN ATOM $V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$

3D ISOTROPIC SHO. $V(r) = \frac{m}{2}\omega^2 r^2 = \frac{m}{2}\omega^2(x^2+y^2+z^2)$

ONTO ANGULAR EQN(S)

$$\hat{L}^2 Y(\theta,\varphi) = \hbar^2 \ell(\ell+1) Y(\theta,\varphi)$$

$$-\cancel{\hbar^2}\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\} = \cancel{\hbar^2}\ell(\ell+1) Y(\theta,\varphi)$$

SEPARATION OF VARIABLES $Y = \Theta(\theta)\Phi(\varphi)$

ALSO SIMULTANEOUS EIGENSTATES OF $\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$

$$\Rightarrow \quad \Phi(\varphi) = e^{im\varphi} \qquad m = \text{integer}$$

$$\Rightarrow -\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\cancel{\Phi} - \frac{m^2}{\sin^2\theta}\Theta\cancel{\Phi}\right\} = \ell(\ell+1)\Theta\cancel{\Phi}$$

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta = \ell(\ell+1)\Theta}$$

THE ONLY FINITE SOLUTIONS ARE
"ASSOCIATED LEGENDRE FUNCTIONS"

$$\Theta(\theta) = P_\ell^m(\cos\theta) \qquad \ell = 0,1,2,3,\ldots$$
$$m = -\ell, -\ell+1, \ldots, +\ell$$

(see QMA) "TRADITIONAL DIFF. EQN.
MT3 APPROACH"

SEE NOTES FOR MORE DETAILS

EX: $P_0^0 = 1$

$P_1^0 = \cos\theta$

$P_1^{\pm1} = \sin\theta$

$P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$

$P_2^{\pm1} = 3\sin\theta\cos\theta$

$P_2^{\pm2} = 3\sin^2\theta$

AND UP TO NORMALISATION (SEE NOTES)

$Y_{00} = 1$

$Y_{10} = \cos\theta \qquad \frac{z}{r}$

$Y_{1\pm1} = \sin\theta\, e^{\pm i\varphi} \qquad \frac{x \pm iy}{r}$

$Y_{20} = 3\cos^2\theta - 1 \qquad \frac{3z^2 - r^2}{r^2} = \frac{2z^2 - x^2 - y^2}{r^2}$

$Y_{2,\pm1} = \sin\theta\cos\theta\, e^{\pm i\varphi} \qquad \frac{z(x \pm iy)}{r^2}$

$Y_{2,\pm2} = \sin^2\theta\, e^{\pm 2i\varphi} \qquad \frac{(x \pm iy)^2}{r^2}$

$\vdots$

NORMALISATION $\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta\, |Y_{lm}|^2 = 1$

EIGENFUNCTIONS OF CENTRAL POTENTIAL

$\psi_{E,l,m}(r,\theta,\varphi) = R_{El}(r)\, Y_{lm}(\theta,\varphi)$

EXPLICIT SOLUTIONS DEPEND ON POTENTIAL

<u>HYDROGEN ATOM</u> $\quad E_n = E_1 \frac{1}{n^2} \quad$ n.. principal quantum nr. $=1,2,3,\dots$

	$\vec{L}^2$	$\hat{L}_z$		DEGENERACY
<u>$n=1$</u>:	$l=0$	$m=0$	$\rightarrow$	1
<u>$n=2$</u>:	$l=0$	$m=0$		$1+3=4$
	$l=1$	$m=-1,0,1$		
<u>$n=3$</u>:	$l=0$	$m=0$		$1+3+5=9$
	$l=1$	$m=-1,0,1$		
	$l=2$	$m=-2,-1,0,1,2$		

$\boxed{\psi_{nlm} = R_{nl}(r)\, Y_{lm}(\theta,\varphi)} \qquad |n,l,m\rangle$

ABSTRACT ANGULAR MOMENTUM OPERATORS

$\hat{J}_{x,y,z}$ HERMITIAN (SELF-ADJOINT)
CAN BE DIFFERENTIAL OPERATORS
LIKE ORBITAL A.M. OPERATORS
$\hat{L}_x, \hat{L}_y, \hat{L}_z$ OR MATRICES

WE ONLY ASSUME NOW GENERAL PROPERTIES

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \ , \ [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

$$[\hat{\vec{J}}^2, \hat{J}_x] = [\hat{\vec{J}}^2, \hat{J}_y] = [\hat{\vec{J}}^2, \cancel{\hbar}\hat{J}_z] = 0$$

$$\hat{\vec{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

SIMULTANEOUS EIGENSTATES ψ_{jm}

$$\psi_{jm} \equiv |\psi_{jm}\rangle = |jm\rangle$$

KET-STATES

$$\hat{\vec{J}}^2 \psi_{jm} = \hbar^2 j(j+1)\psi_{jm}$$

$$\hat{J}_z \psi_{jm} = \hbar m \psi_{jm}$$

$$[\hat{\vec{J}}^2, \hat{J}_z] = 0$$

j, m REAL SINCE $\hat{\vec{J}}^2, \hat{J}_z$ ARE HERMITIAN

GOALS: (1) DERIVE ALLOWED VALUES OF j, m USING GENERAL PROPERTIES

(2) FIND A BETTER WAY TO CALCULATE $\hat{\vec{L}}^2, \hat{L}_z$ EIGEN FUNCTIONS $Y_{lm}(\theta, \varphi)$

$$\hat{\vec{L}}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}, \quad \hat{L}_z Y_{lm} = \hbar m Y_{lm}$$

FOR ORBITAL A.M. l AND m INTEGER

$l = 0, 1, 2, \ldots$ AND $\underbrace{m = l, l-1, l-2, \ldots, -l}_{2l+1}$

FROM (1) WE WILL FIND ANOTHER POSSIBILITY

$j = 0, 1, 2, \ldots$

OR $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

(1) CONSIDER ABSTRACT A.M. EIGENSTATE

$\psi = \psi_{jm}$ (NORMALISED)

$$\langle \hat{\vec{J}}^2 \rangle = \langle \psi_{jm} | \hat{\vec{J}}^2 \psi_{jm} \rangle = \hbar^2 j(j+1) \underbrace{\langle \psi_{jm} | \psi_{jm} \rangle}_{=1}$$

$$= \hbar^2 j(j+1)$$

$$= \underbrace{\langle \hat{J}_x^2 \rangle}_{\geq 0} + \underbrace{\langle \hat{J}_y^2 \rangle}_{\geq 0} + \underbrace{\langle \hat{J}_z^2 \rangle}_{m^2 \hbar^2}$$

$$\Rightarrow \hbar^2 j(j+1) \geq m^2 \hbar^2$$

$$\boxed{j(j+1) \geq m^2}$$ m IS BOUNDED!

A.M. RAISING / LOWERING OPERATORS

RAISING OP.: $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$

LOWERING OP.: $\hat{J}_- = \hat{J}_x - i\hat{J}_y$

$$(\hat{J}_+)^\dagger = (\hat{J}_x + i\hat{J}_y)^\dagger = \hat{J}_x - i\hat{J}_y = \hat{J}_-$$

PROPERTIES:

$$[\hat{\vec{J}}^2, \hat{J}_\pm] = 0 \Rightarrow \hat{\vec{J}}^2(\hat{J}_\pm \psi_{jm}) =$$

$$= \hat{J}_\pm \hat{\vec{J}}^2 \psi_{jm} = \hbar^2 j(j+1)(\hat{J}_\pm \psi_{jm})$$

$\Rightarrow \hat{J}_\pm \psi_{jm}$ has same $\hat{\vec{J}}^2$ EIGENVALUE

I $[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x] + i[\hat{J}_z, \hat{J}_y]$

$$= i\hbar\hat{J}_y + i(-i\hbar)\hat{J}_x = \hbar\hat{J}_+$$

SIMILAR: $[\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-$

($\hat{J}_+, \hat{J}_-$ ARE SIMILAR TO $\hat{a}^\dagger, \hat{a}$ OF S.H.O

RECALL $[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger \Rightarrow \hat{a}^\dagger\psi_n \propto \psi_{n+1}$.

S.H.O. EIGENSTATES)

SO IF ψ_{jm} IS NORMALISED THEN

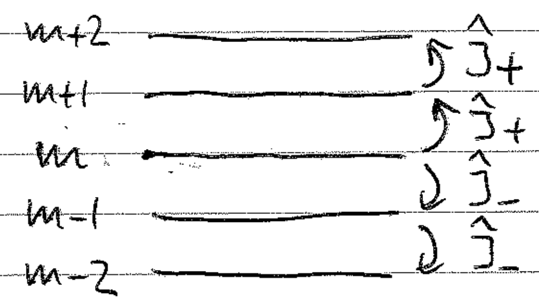
$$\boxed{\psi_{jm+1} = \frac{1}{\hbar\sqrt{j(j+1)-m(m+1)}}\,\hat{J}_+\,\psi_{jm}}$$

PROPERLY NORMALISED

SIMILARLY

$$\boxed{\psi_{jm-1} = \frac{1}{\hbar\sqrt{j(j+1)-m(m-1)}}\,\hat{J}_-\,\psi_{jm}}$$

j FIXED



SINCE m IS BOUNDED $j(j+1) \geq m^2$

THIS PROCESS OF GENERATING NEW STATES MUST TERMINATE!

SO THERE MUST BE m_{max} AND m_{min} SUCH THAT

$$\hat{J}_+ \psi_{jm_{max}} = 0$$

$$\hat{J}_- \psi_{jm_{min}} = 0$$

ALSO $m_{max} - m_{min} =$ <u>INTEGER</u>

FROM (I) $\quad \hat{J}_z \hat{J}_+ = \hat{J}_+ \hat{J}_z + \hbar \hat{J}_{\cancel{z}+}$

$$\Rightarrow \hat{J}_z(\hat{J}_+ \psi_{jm}) = \hat{J}_+ (\underbrace{\hat{J}_z \psi_{jm}}_{\hbar m \psi_{jm}}) + \hbar(\hat{J}_+ \psi_{jm})$$

$$= \hbar(m+1)(\hat{J}_+ \psi_{jm})$$

↑

INCREASED $\hat{J}_z$ EIGENVALUE BY $\hbar$

$m \to m+1$

SO $\boxed{\hat{J}_+ \psi_{jm} \propto \psi_{jm+1}}$

SIMILARLY $\boxed{\hat{J}_- \psi_{jm} \propto \psi_{jm-1}}$

<u>NORMALISATION</u> (CAN BE SKIPPED ON A FIRST READ RESULT USED LATER)

$$\langle \hat{J}_+ \psi_{jm} | \hat{J}_+ \psi_{jm} \rangle = \text{NORM SQUARED OF } \hat{J}_+\psi_{jm}$$

$$= \langle \psi_{jm} | (\hat{J}_+)^\dagger \hat{J}_+ \psi_{jm} \rangle = \langle \psi_{jm} | \hat{J}_- \hat{J}_+ \psi_{jm} \rangle$$

$$\hat{J}_- \hat{J}_+ \psi_{jm} = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)\psi_{jm}$$

$$= (\hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y])\psi_{jm}$$

$$= (\hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z)\psi_{jm}$$

$$= (\hbar^2 j(j+1) - \hbar^2 m^2 - \hbar(\hbar m))\psi_{jm}$$

$$= \hbar^2[j(j+1) - m(m+1)]\psi_{jm}$$

$$= \hbar^2[j(j+1) - m(m+1)] \underbrace{\langle \psi_{jm} | \psi_{jm} \rangle}_{=1}$$

see previous page

$$\hat{J}_+ \psi_{jm_{max}} = 0$$

$$\Rightarrow \hat{J}_- \hat{J}_+ \psi_{jm_{max}} = 0 = (\hat{\vec{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z)\psi_{jm_{max}}$$

$$= \hbar^2 [j(j+1) - m_{max}(m_{max}+1)]\psi_{jm_{max}} = 0$$

$$\Rightarrow j(j+1) = m_{max}(m_{max}+1)$$

$$\Rightarrow \boxed{m_{max} = j}$$

(WITHOUT LOSS OF GENERALITY WE CHOOSE $j \geq 0$)

SIMILARLY: $\hat{J}_- \psi_{jm_{min}} = 0$

$$\hat{J}_+ \hat{J}_- \psi_{jm_{min}} = \ldots = \hbar^2 [j(j+1) - m_{min}(m_{min}-1)]\psi_{jm_{min}} = 0$$

$$\Rightarrow j(j+1) = m_{min}(m_{min}-1)$$

$$\Rightarrow \boxed{m_{min} = -j}$$

(WE CAN DISCARD 2^{nd} SOLUTION $(j+1)$ BECAUSE $m_{max} \geq m_{min}$)

- ALLOWED VALUES OF m

$$\boxed{m = j, j-1, j-2, \ldots, -j}$$

$$m_{max} - m_{min} = 2j = \text{INTEGER!}$$

<u>TWO POSSIBILITIES</u>

$j = 0, 1, 2, 3$ <u>INTEGER</u>

OR $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ <u>HALF-INTEGER</u>

DIFFERENTIAL OPERATORS

$j = 0, 1, 2, 3, 4 ..$ $\rightarrow$ ORBITAL A.M. $\hat{\vec{L}}$, $j = \ell$

$$\hat{\vec{L}}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$\hat{L}_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$$m = \underbrace{\ell, \ell-1, \ell-2, \dots, -\ell}_{}$$

$2\ell+1$ POSSIBLE EIGENVALUES OF $\hat{L}_z$ (ODD NUMBER)

$\rightarrow$ SPIN A.M. (MATRICES)

$j = 0$ SCALAR FIELD E.g. Higgs

$j = 1$ PHOTONS, GLUONS, $W_\pm$, Z_0

$j = 2$ GRAVITON

$j = \frac{1}{2}, \frac{3}{2}, \dots$ $\rightarrow$ SPIN A.M. (MATRICES, CANNOT BE REALISED AS DIFF. OPERATORS)

$j = \frac{1}{2}$, $m = +\frac{1}{2}, -\frac{1}{2}$: SPIN UP/DOWN

electrons, quarks, ...

$j = \frac{3}{2}$, $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

e.g. Δ^{++} baryon $(\overset{\uparrow}{u}\overset{\uparrow}{u}\overset{\uparrow}{u})$

3 up quarks

$$m = \underbrace{j, j-1, \dots, -j}_{}$$

$2j+1$ = EVEN NUMBER

$\Downarrow$

THAT'S HOW PEOPLE DISCOVERED SPIN

ORBITAL A.M. EIGENFUNCTIONS Y_{lm}
(SPHERICAL HARMONICS) IN OPERATOR APPROACH

$$\hat{\vec{L}}^2 \, Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1)\, Y_{lm}(\theta,\varphi)$$

$$\hat{L}_z Y_{lm}(\theta,\varphi) = \hbar m \, Y_{lm}(\theta,\varphi) \,, \quad \hat{L}_z = -i\hbar \frac{\partial}{\partial\varphi}$$

EIGENFUNCTIONS OF $\hat{L}_z$: $e^{im\varphi}$

m INTEGER $\Rightarrow$ l INTEGER

$$m = l, l-1, l-2, \dots, -l$$

LOWERING/RAISING OPERATORS

$$\hat{L}_x = i\hbar\left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_y = i\hbar\left(-\cos\varphi \frac{\partial}{\partial\theta} + \cot\theta \sin\varphi \frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \hbar\Big((i\sin\varphi + \cos\varphi)\frac{\partial}{\partial\theta} + i\cot\theta\,\underbrace{(\cos\varphi + i\sin\varphi)}_{e^{i\varphi}}\frac{\partial}{\partial\varphi}\Big)$$

$$\Rightarrow \boxed{\hat{L}_+ = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)}$$

$$\boxed{\hat{L}_- = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)}$$

GENERAL PROPERTIES

$$\hat{L}_\pm Y_{lm} \propto Y_{l\,m\pm 1}$$

ALSO: $m=l$ is maximal value of m
$m=-l$ is minimum value of m

$$\Rightarrow \quad \hat{L}_+ Y_{ll}(\theta,\varphi) = 0$$

ALSO $Y_{ll}(\theta,\varphi) = f(\theta)\, e^{il\varphi}$

SINCE $\hat{L}_z Y_{ll} = \hbar l\, Y_{ll}$

$$\hat{L}_+ \left(f(\theta) e^{il\varphi}\right)$$

$$= \hbar e^{i\varphi}\left(\frac{d f(\theta)}{d\theta} e^{il\varphi} + i\cot\theta\, f(\theta) \underbrace{\frac{\partial e^{il\varphi}}{\partial\varphi}}_{= il e^{il\varphi}}\right)$$

$$= \hbar e^{i\varphi}\left(\frac{df}{d\theta} - l\cot\theta\, f(\theta)\right) e^{il\varphi} = 0$$

$$\Rightarrow \quad \frac{df}{d\theta} = l\,\frac{\cos\theta}{\sin\theta}\, f(\theta)$$

$$f(\theta) = N_l (\sin\theta)^l, \quad f' = N_l \cos\theta\,(\sin\theta)^{l-1} \times l$$

✓

SO $\boxed{Y_{ll}(\theta,\varphi) = N_l (\sin\theta)^l e^{il\varphi}}$

Y_{lm} FOR $m<l$ OBTAINED BY ACTING WITH $\hat{L}_-$!

EXAMPLE $\quad Y_{11} = \mathcal{N}_1 \sin\theta\, e^{i\varphi}$

$$Y_{1,0} = \frac{1}{\hbar\sqrt{\underbrace{l(l+1)-m(m-1)}_{l=m=1}}}\hat{L}_- Y_{11}$$

$$= \frac{1}{\hbar\sqrt{2}}\hbar e^{-i\varphi}\mathcal{N}_1\left(-\frac{\partial}{\partial\theta}(\sin\theta e^{i\varphi}) + i\cot\frac{\partial(\sin\theta e^{i\varphi})}{\partial\varphi}\right)$$

$$= \frac{\mathcal{N}_1}{\sqrt{2}}\cancel{e^{-i\varphi}}\underbrace{\left(-\cos\theta\cancel{e^{i\varphi}} + i\frac{\cos\theta}{\sin\theta}\sin\theta\, i\cancel{e^{i\varphi}}\right)}_{=-2\cos\theta}$$

$$= -\sqrt{2}\,\mathcal{N}_1\cos\theta$$

SEE NOTES $\quad \mathcal{N}_1 = -\left(\frac{3}{8\pi}\right)^{\frac{1}{2}}$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad \checkmark$$

- $Y_{1,-1} \sim \hat{L}_- Y_{10}$
- NOTE: LIKE THIS WE CAN GET ALL $Y_{lm}(\theta,\varphi)$

FROM $Y_{ll} = \mathcal{N}_l(\sin\theta)^l e^{il\varphi}$.

- ONE CAN CHECK THAT ALL THIS EIGENFUNCTIONS ARE INDEED EIGENFUNCTIONS OF

$$\hat{\vec{L}}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

2^{nd} ORDER DIFF. OPERATOR!

WITH EIGENVALUES $\hbar^2 l(l+1)$

• WHY DO WE NEED MATRIX MECHANICS FOR $j=\frac{1}{2}$?

OR WHAT GOES WRONG FOR ORBITAL A.M. WAVEFUNCTIONS FOR $\ell=\frac{1}{2}$?

$\ell=m=\frac{1}{2}$: $Y_{\frac{1}{2},\frac{1}{2}} \sim (\sin\theta)^{1/2} e^{i\varphi/2}$ → not periodic under $\varphi \to \varphi + 2\pi$

STILL $\hat{L}_+ Y_{\frac{1}{2}\frac{1}{2}} = 0$

$$Y_{\frac{1}{2},-\frac{1}{2}} \sim \hat{L}_- Y_{\frac{1}{2}\frac{1}{2}} = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}\left(\sqrt{\sin\theta}\, e^{i\varphi/2}\right) + i\cot\theta \frac{\partial}{\partial\varphi}\left(\sqrt{\sin\theta}\, e^{i\varphi/2}\right)\right)$$

$$= \hbar e^{-i\varphi}\left(-\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}} e^{i\varphi/2} - \frac{1}{2}\frac{\cos\theta}{\sin\theta}\sqrt{\sin\theta}\, e^{i\varphi/2}\right)$$

$$= -\hbar e^{-i\varphi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}$$

→ singular for $\theta = 0, \pi$!

• THESE CANNOT BE MEANINGFUL WAVEFUNCTIONS

• NEED TO RESORT TO <u>MATRIX QUANTUM MECH.</u>

E.G. $j=\frac{1}{2}$ SPIN $\frac{1}{2}$ $m=\frac{1}{2},-\frac{1}{2}$

$\hat{\vec{J}} = \vec{S} = \frac{\hbar}{2}\vec{\sigma}$ → 2×2 matrices

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PAULI MATRICES $\sigma_{x,y,z}$

• VECTOR MODEL OF A.M.

$$\hat{\vec{J}}^2 \psi_{jm} = \hbar^2 j(j+1)\psi_{jm} \ , \quad |\vec{J}| = \hbar\sqrt{j(j+1)}$$

$$\hat{J}_z \psi_{jm} = \hbar m \ \psi_{jm}$$

$\hat{\vec{J}}^2, \hat{J}_z$ can be measured simultaneously

$$\boxed{\Delta \hat{\vec{J}}^2 = \Delta \hat{J}_z = 0}$$

FOR $\psi = \psi_{jm}$: $\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$

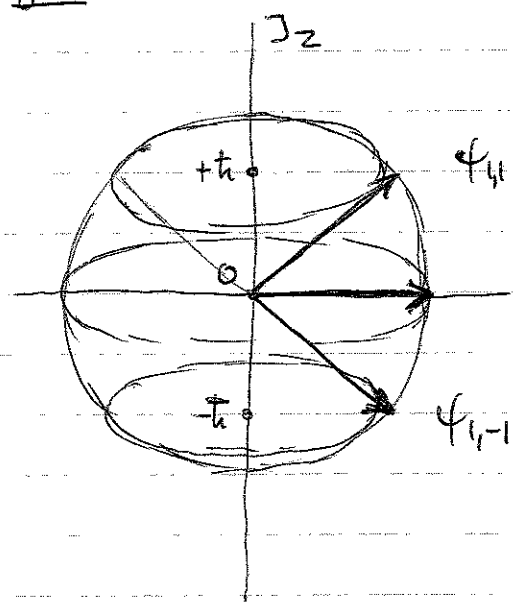
$$\langle \hat{J}_x^2 + \hat{J}_y^2 \rangle = \langle \hat{\vec{J}}^2 - \hat{J}_z^2 \rangle = \hbar^2 [j(j+1) - m^2]$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 = \hbar^2 [j(j+1) - m^2]$$

THIS IS SMALLEST FOR $m = \pm j$

$$\rightarrow (\Delta J_x)^2 + (\Delta J_y)^2 = \hbar^2 j$$

$j=1$ $\quad |\vec{J}| = \hbar\sqrt{2}$



IF $\hat{\vec{J}}$ IS MEASURED RESULT MOST LIKELY TO BE A VECTOR INSIDE THE CONE:

only $\vec{J}^2, J_z$ have definite values

$j = \frac{1}{2}$ $\quad m = \pm\frac{1}{2}$