WEEK 11 /L1

SPIN + SPIN WAVEFUNCTIONS

SPIN j: S_x, S_y, S_z $(2j+1)\times(2j+1)$ dim'l matrices

STATES ARE $2j+1$ dim'l vectors

$$\chi = \begin{pmatrix} c_j \\ c_{j-1} \\ \vdots \\ c_{-j} \end{pmatrix}$$

$m = j, j-1, \dots, -j$

$|c_m|^2$.. Probability to find $\hbar m$ in measurement of $\hat{J}_z$

SPIN $j=\frac{1}{2}$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} \qquad a, b \text{ complex}$$

NORMALISATION $\chi^+\chi = |a|^2 + |b|^2 = 1$

EXPECTATION VALUES: $\langle \hat{A} \rangle = \chi^+ \underbrace{A}_{2\times 2 \text{ matrix}} \chi$

E.g. $A = S_x$

$$\langle S_x \rangle = \chi^+ S_x \chi = (a^* \;\; b^*)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{\hbar}{2}(a^* \;\; b^*)\begin{pmatrix} b \\ a \end{pmatrix} = \frac{\hbar}{2}(a^* b + b^* a)$$

$$= \hbar \,\mathrm{Re}(a^* b)$$

Q IF WE MEASURE S_z (USING FOR EXAMPLE S.G. EXPERIMENT) WHAT ARE POSSIBLE OUTCOMES

Q AFTER MEASUREMENT W.F. COLLAPSES TO ONE OF THE EIGENSTATES, WHAT ARE THEY

$$S_z \chi = \lambda \chi$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ -b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$ 2 SOLUTIONS

$$\Rightarrow \cancel{b \neq 0}, \quad \frac{\hbar}{2} a = \lambda a$$

$$-\frac{\hbar}{2} b = \lambda b$$

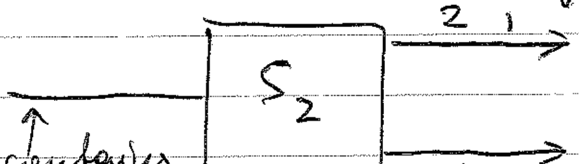
<u>EIGENVALUE</u> <u>EIGENVECTOR/STATE</u>

<u>2 SOLN.</u>: $b = 0, \ \lambda = \frac{\hbar}{2}, \ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a = 0, \ \lambda = -\frac{\hbar}{2}, \ \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>MEASUREMENT</u>

$\frac{\hbar}{2}$, W.F. COLLAPSES TO $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Beam contains of atoms with all possible spin states

$-\frac{\hbar}{2}$, W.F. COLL. $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>EXPANSION THEOREM</u>

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\chi_+ + b\chi_-$$

GENERAL WAVEFUNCTION OF e^-

ENERGY EIGENSTATES : 4 QUANTUM NUMBERS

$$n\,,\; l\,,\; m\,,\; m_s = \pm\tfrac{1}{2}$$

$n \downarrow r$, $l \downarrow \theta$, $m \downarrow \varphi$

m_s ← Spin (internal degree of freedom

$$R_{nl}(r)\, Y_{lm}(\theta,\varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad m_s = +\tfrac{1}{2}$$

$$R_{nl}(r)\, Y_{lm}(\theta,\varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad m_s = -\tfrac{1}{2}$$

GENERAL W.F. OF e^- $\boxed{\Psi = \Psi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Psi_-(r,\theta,\varphi,t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

PAULI EQUATION

$$\hat{H} = \frac{1}{2m}(\hat{\vec{p}} + e\hat{\vec{A}})^2 + V(r) + 2\frac{\mu_B}{\hbar}\vec{B}\cdot\underbrace{\vec{S}}_{2\times 2 \text{ matrix}}$$

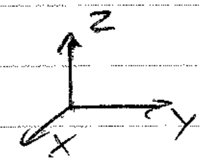
$$\cong \frac{\hat{\vec{p}}^2}{2m} + \frac{\mu_B}{\hbar}\vec{B}\cdot(\hat{\vec{L}} + 2\vec{S}) + V(r)$$

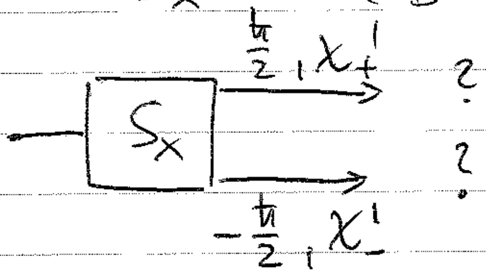
CAN BE DERIVED BY TAKING NON-RELATIVISTIC LIMIT OF DIRAC EQUATION $(v \ll c)$

ROTATED SPIN OPERATORS

WHAT IF WE CONSIDER S.G. EXPERIMENT ROTATED BY 90° AROUND y-AXIS

$\Rightarrow S_x \quad (\vec{B} = (B_x, 0, 0))$





$$S_x \chi' = \frac{\hbar}{2} \lambda \chi' \qquad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \chi' = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = \lambda a, \quad a = \lambda b \quad \Rightarrow \quad b = \lambda(\lambda b) = \lambda^2 b$$

$$\Rightarrow \lambda^2 = 1, \quad \lambda = \pm 1$$

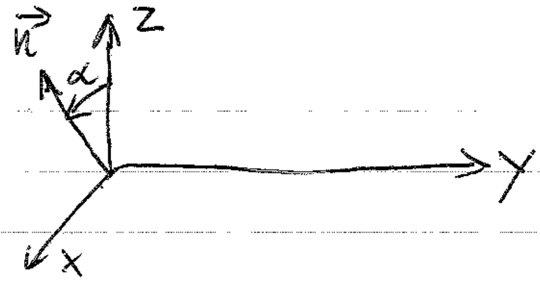
$$\left.\begin{aligned} \lambda = 1: \ a = b \qquad & \chi' = \begin{pmatrix} a \\ a \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda = -1: \ b = -a \qquad & \chi' = \begin{pmatrix} a \\ -a \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}\right\} \text{normalised}$$

S_x: EIGENVALUES EIGENSTATES

$$\frac{\hbar}{2} \qquad \chi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$$

$$-\frac{\hbar}{2} \qquad \chi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+ - \chi_-)$$

ARBITRARY ANGLE



ROTATED S.G.

$\vec{n} = (\sin\alpha, 0, \cos\alpha)$, $\vec{n}^2 = 1$

$$S_{\vec{n}} = \vec{n}\cdot\vec{S} = \sin\alpha\, S_x + 0 + \cos\alpha\, S_z = \frac{\hbar}{2}\begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix}$$

EIGENVALUES: $\pm\frac{\hbar}{2}$

$$S_n \chi'' = \pm\frac{\hbar}{2}\chi''$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\cos\alpha + b\sin\alpha\\ a\sin\alpha - b\cos\alpha\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}a\\ b\end{pmatrix}$$

$+\frac{\hbar}{2}$: $a\cos\alpha + b\sin\alpha = a$ $\qquad b\sin\alpha = a(1-\cos\alpha)$

$a\sin\alpha - b\sin\alpha = b$

$$\frac{b}{a} = \frac{(1-\cos\alpha)}{\sin\alpha} = \frac{1-\cos^2(\frac{\alpha}{2}) + \sin^2(\frac{\alpha}{2})}{2\cos(\frac{\alpha}{2})\sin(\frac{\alpha}{2})} = \frac{\cancel{2}\sin^{\cancel{2}}(\frac{\alpha}{2})}{\cancel{2}\cancel{\sin(\alpha/2)}\cos(\frac{\alpha}{2})}$$

$$\frac{b}{a} = \frac{\sin(\frac{\alpha}{2})}{\cos(\frac{\alpha}{2})} \qquad \chi''_+ = \begin{pmatrix}\cos(\frac{\alpha}{2})\\ \sin(\frac{\alpha}{2})\end{pmatrix}$$

$$-\frac{\hbar}{2}:\ \frac{b}{a} = \frac{-1-\cos\alpha}{\sin\alpha} = \frac{-2\cos^2(\frac{\alpha}{2})}{2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})} = -\frac{\cos(\frac{\alpha}{2})}{\sin(\frac{\alpha}{2})}$$

$$\chi''_- = \begin{pmatrix}\sin(\alpha/2)\\ -\cos(\alpha/2)\end{pmatrix}$$

ROTATION BY α IN z-x PLANE

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}\sin\alpha\\ 0\\ \cos\alpha\end{pmatrix} \quad \text{VECTOR}$$

ROTATION OF SPIN W.F.

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\cos(\alpha/2)\\ \sin(\alpha/2)\end{pmatrix}, \quad \chi_- = \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}\sin(\alpha/2)\\ -\cos(\alpha/2)\end{pmatrix}$$

<u>2π ROTATION</u> VECTOR $\vec{x} \to \vec{x}$

$$\chi_+''(\alpha = 2\pi) = \begin{pmatrix} \cos\pi \\ \sin\pi \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\chi_+ \; !$$

<u>COMMENT</u> - SPINOR WAVEFUNCTIONS (SPINORS) CHANGE SIGN UNDER 2π ROTATION

- NOT SINGLE VALUED

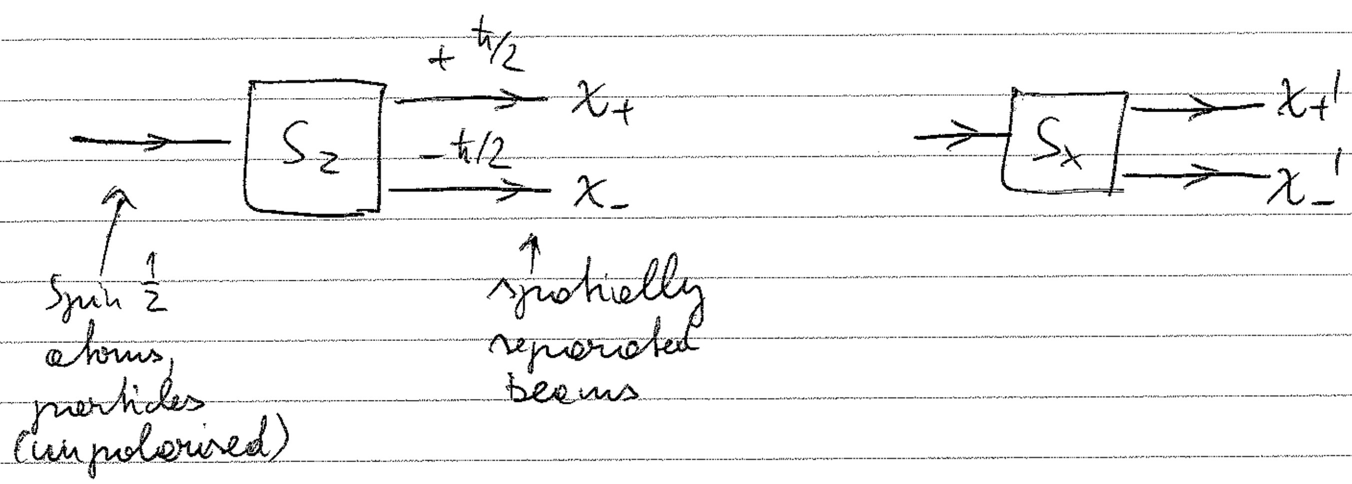
- PHYSICAL QUANTITIES LIKE EXPECTATION VALUES NOT AFFECTED

$$\langle A \rangle = \chi^\dagger A \chi \to (-\chi^\dagger) A (-\chi)$$
$$= \chi^\dagger A \chi$$

THE TRIPLE STERN GERLACH EXPERIMENT

STERN GERLACH EXP.: → MEASURES COMPONENT OF SPIN

→ PREPARES ENSEMBLES OF STATES WITH DEFINITE ~~??~~ VALUE OF SPIN COMPONENT



ILLUSTRATES MEASUREMENT POSTULATE : ONLY TWO POSSIBLE OUTCOMES FOR SPIN $j = \frac{1}{2}$

$+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$

NOTE ~~PARTICLES MUST TAKE~~ ONE OR THE OTHER WAY!

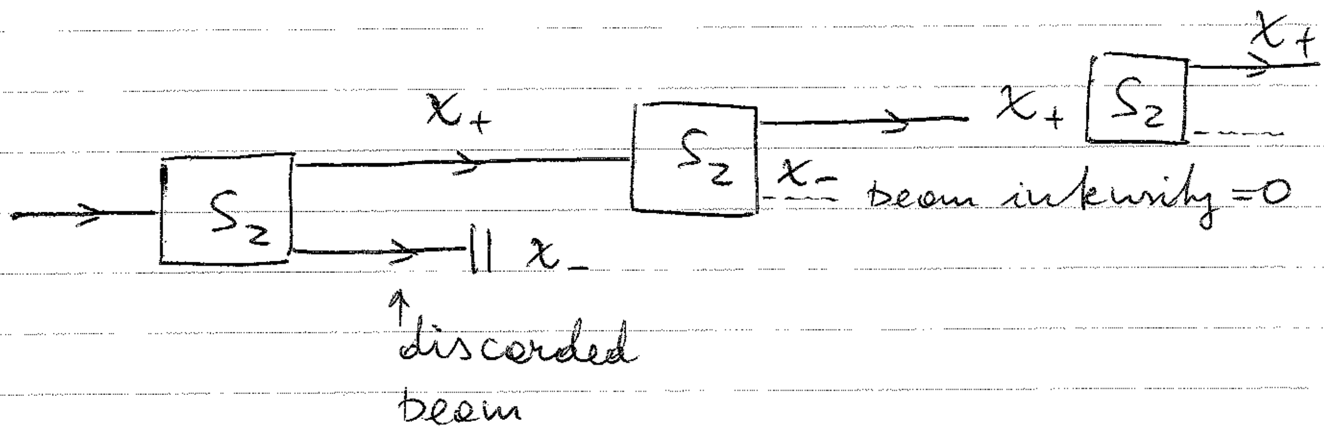
BUT THIS IS TRUE FOR MEASUREMENT OF

S_z AND S_x AND S_n

INCOMING PARTICLE BEAM IS IN "INCOHERENT MIXTURE"

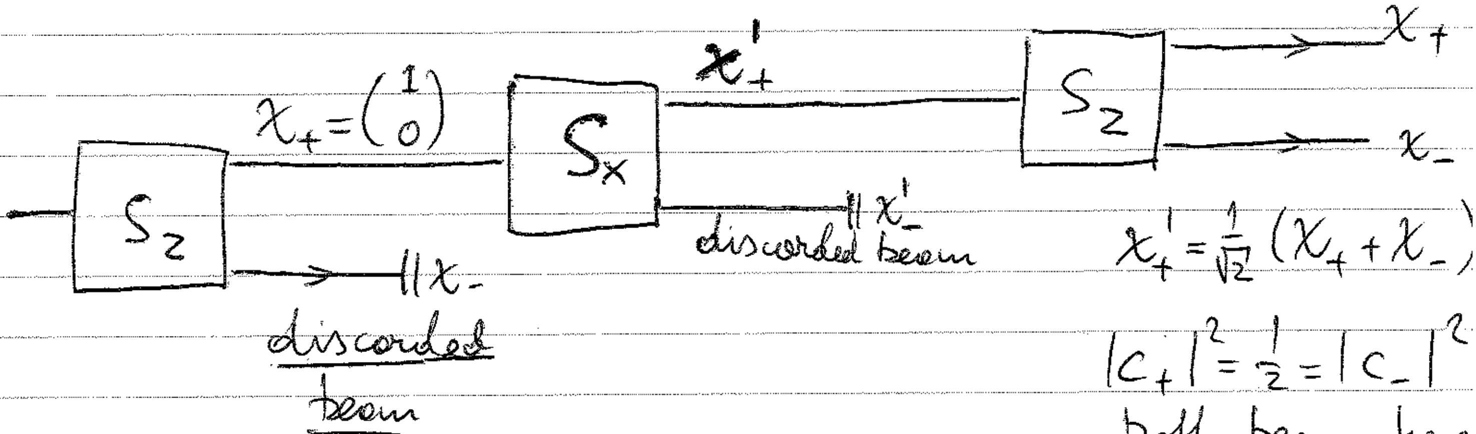
OF STATE χ_+ AND χ_-

DOUBLE STERN GERLACH "WARM-UP"



W.F. collapse, system prepared in state χ_+

TRIPLE STERN GERLACH



$\chi'_+ = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$

$|c_+|^2 = \frac{1}{2} = |c_-|^2$

both beams have same intensity

$$\chi'_+ = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$$

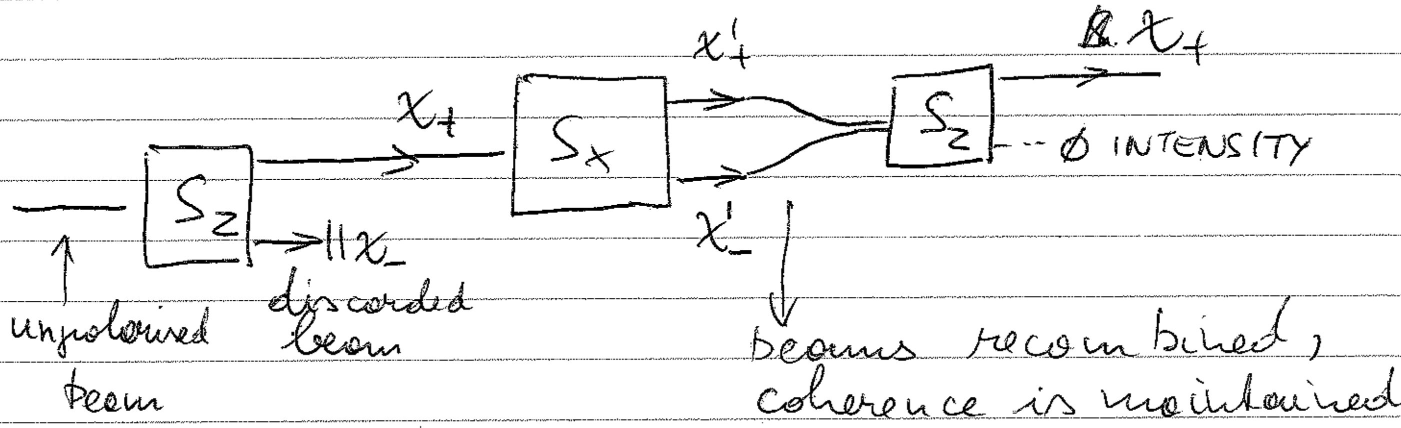
$$\chi'_- = \frac{1}{\sqrt{2}}(\chi_+ - \chi_-)$$

$$\Rightarrow \chi_+ = \frac{1}{\sqrt{2}}(\chi'_+ + \chi'_-)$$

$$\chi_- = \frac{1}{\sqrt{2}}(\chi'_+ - \chi'_-)$$

- INTENSITY OF FINAL BEAM χ_+ (OR χ_-) COMPARED TO INCOMING BEAM $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
- <u>REGENERATION</u>: IN LAST STAGE χ_- reappears although IT WAS DISCARDED IN THE FIRST STAGE

MODIFIED TRIPLE S.G.



$$\cancel{\chi'_+ + \chi'_- = \sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\chi'_+ + \frac{1}{\sqrt{2}}\chi'_- = \chi_+$$

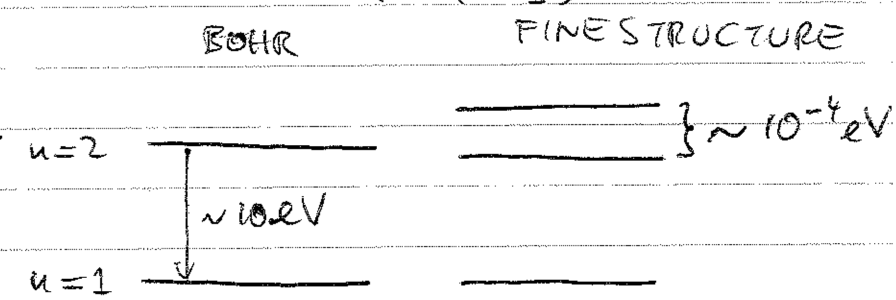
THIS SHOWS THAT THE RESULT OF ~~THE~~ 1ST EXPERIMENT IS NOT A RESULT OF A DISTURBANCE INSIDE $\boxed{S_x}$. THE EFFECT IS DUE TO THE MEASUREMENT AFTER $\boxed{S_x}$ THAT LEADS TO A COLLAPSE OF THE W.F.!

SPIN-ORBIT COUPLING AND ADDITION OF A.M.

EVEN FOR $\vec{B}=0$ (EXTERNAL) THE ENERGIES OF SPECTRAL LINES ARE NOT EXACTLY AS PREDICTED BY BOHR

GROSS STRUCTURE: $E_n = \frac{E_1}{n^2}$ (independent of l, m, m_s)

FINE STRUCTURE



- ATTEMPT TO EXPLAIN FINE STRUCTURE LED GOUDSMIT-UHLENBECK TO PROPOSE SPIN.

SPIN ORBIT-COUPLING

- proton produces electric field: $\vec{E} = \frac{e\vec{x}}{4\pi\epsilon_0 r^3}$

$r = |\vec{x}|$

- electron e^- moves with velocity $\vec{v}$

ACCORDING TO SPECIAL RELATIVITY THE e^- FEELS IN ITS REST FRAME A MAGNETIC FIELD

$$\vec{B} \simeq -\frac{1}{c^2}\vec{v}\times\vec{E} = \left(-\frac{\vec{p}}{c^2 m}\right)\times\left(e\frac{\vec{x}}{4\pi\epsilon_0 r^3}\right)$$

$$= \frac{e}{c^2 m 4\pi\epsilon_0 r^3}\vec{L}$$

$$H_{SO} = -\vec{\mu}\cdot\vec{B}$$

in restframe A.M. of electron only due to spin $\vec{S}$!

$$\vec{\mu} = -\underbrace{g_s}_{=\,2}\ \underbrace{\frac{\mu_B}{\hbar}}_{=\,\frac{e}{2m}}\ \vec{S} \approx -\frac{e}{m}\vec{S}$$

$$= \frac{e^2}{c^2 m_e^2 4\pi\epsilon_0 r^3}\ \vec{L}\cdot\vec{S}$$

- **NOTE** THIS HAS TO BE CORRECTED, BECAUSE e^- IS IN ACCELERATED MOTION (THOMAS PRECESSION)
  .. LONG STORY .. **FACTOR $\frac{1}{2}$**

$$\Rightarrow \hat{H}_{SO} = \underbrace{\frac{e^2}{8\pi\epsilon_0 m_e^2 c^2 r^3}}\ \hat{\vec{L}}\cdot\vec{S}$$

APPROXIMATE THIS BY EXPECTATION VALUE

$\underline{l \neq 0}$ $$\left\langle \frac{1}{r^3}\right\rangle_{n,l} = \frac{2}{a_0^3 n^3 l(l+1)(2l+1)} \equiv \frac{c_{n,l}}{a_0^3}$$

$$\hat{H}_{SO} \simeq \underbrace{\frac{e^2\hbar^2 c_{n,l}}{8\pi\epsilon_0 c^2 m_e^2 a_0^3}}_{10^{-5}-10^{-4}\,eV\ \ll\ |E_1|}\left(\frac{1}{\hbar^2}\hat{\vec{L}}\cdot\vec{S}\right)$$

BOHR RADIUS

$$\hat{H} = \hat{H}_0 + \hat{H}_{SO}$$

IF $\vec{B}_{EXT} \neq 0$ $\qquad$ Zeeman effect

$$+ \frac{\mu_B}{\hbar} \vec{B}_{EXT} \cdot (\hat{\vec{L}} + 2\vec{S})$$

HAMILTONIAN WITH CENTRAL POTENTIAL $V = -\frac{e^2}{4\pi\epsilon_0 r}$

RECALL $\{\hat{H}_0, \hat{\vec{L}}^2, \hat{L}_z, \hat{\vec{S}}^2, S_z\}$ complete set of commuting operators

$\Rightarrow$ simultaneous eigenstates

$\psi_{n l m m_s}$

HOWEVER $\hat{H}_{SO} \propto (\hat{L}_x S_x + \hat{L}_y S_y + \hat{L}_z S_z)$

DOES NOT COMMUTE WITH $\hat{L}_z$ OR $\hat{S}_z$

ONLY WITH $\hat{L}_z + \hat{S}_z$!

$\Rightarrow$ **INTRODUCE** TOTAL A.M. OF ELECTRON (ADDITION OF A.M.)

$$\boxed{\hat{\vec{J}} = \hat{\vec{L}} + \vec{S}}$$

EIGENVALUES: $\{j, m_j\}$ $\quad \{l, m\}$ $\quad \{s, m_s\}$

OP.	EIGENVALUE
$\hat{\vec{J}}^2$	$\hbar^2 j(j+1)$
$\hat{J}_z$	$\hbar m_j$
$\hat{\vec{L}}^2$	$\hbar^2 l(l+1)$
$\hat{L}_z$	$\hbar m$
$\vec{S}^2$	$s(s+1)\hbar^2$
S_z	$\hbar m_s$

$$\hat{\vec{J}}^2 = \hat{\vec{L}}^2 + \vec{S}^2 + 2\hat{\vec{L}} \cdot \vec{S}$$

$$\hat{\vec{L}} \cdot \vec{S} = \frac{1}{2}\left[\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2\right]$$

$\hat{\vec{L}}^2, \vec{S}^2$ COMMUTE WITH ALL COMPONENTS OF $\hat{\vec{J}} = \hat{\vec{L}} + \vec{S}$ ✓

$$\Rightarrow [\hat{\vec{J}}^2, \hat{\vec{L}}^2] = [\hat{\vec{J}}^2, \vec{S}^2] = 0$$

HENCE $\hat{\vec{J}}^2, \vec{L}^2, \vec{S}^2$ ALL COMMUTE WITH $\hat{H}$

AND $[\hat{H}, \hat{J}_z] = 0$.

$\{\hat{H}, \hat{\vec{J}}^2, \hat{\vec{L}}^2, \vec{S}^2, \hat{J}_z\} \equiv$ complete set of commuting operators

<u>A.M. EIGENSTATES</u>

$\{\underset{l,\ m}{\underline{\hat{\vec{L}}^2}, \hat{L}_z}, \underset{s=\frac{1}{2},\ m_s=\pm\frac{1}{2}}{\underline{\vec{S}^2}, S_z}\}$	$\{\hat{\vec{J}}^2, J_z, \underline{\hat{\vec{L}}^2}, \underline{\vec{S}^2}\}$ commute with $\hat{H}$
$Y_{lm}\chi_\pm \equiv \lvert l m s m_s\rangle$	$\lvert j m_j l s\rangle$

<u>GOAL</u> EXPRESS $\lvert j m_j l_s\rangle$ AS LINEAR COMB. OF

$$\lvert l m s m_s\rangle = Y_{lm}\chi_\pm$$

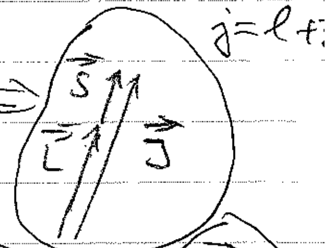
$$m = +l, l-1, \ldots$$
$$m_s = +\tfrac{1}{2}, -\tfrac{1}{2}$$

$$\hat{J}_z = \hat{L}_z + S_z$$

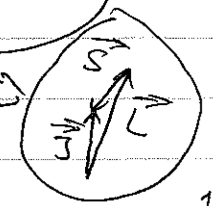
$$\Rightarrow \boxed{m_{j,max} = l + \tfrac{1}{2} = j_{max}}$$

$Y_{\ell m}\chi_{\pm}$ .... $\underline{(2\ell+1)\cdot 2}$ STATES

$j=\ell+\frac{1}{2}$ ... $(2j+1) = 2\ell+2$ STATES $\Rightarrow$



$j=\ell-\frac{1}{2}$ ... $(2j+1) = \underline{2\ell}$ STATES



$\underline{\underline{2(2\ell+1)\text{ STATES}}}$ ✓

- $j=m_j=\ell+\frac{1}{2}$ $\underline{\text{STATE}}$ $\quad |j\, m_j\, \ell s\rangle \equiv \Phi_{jm_j}$

$$\boxed{\Phi_{jj} = \cancel{Y_{\ell\ell}}\, Y_{\ell\ell}\,\chi_+}$$

unique state with maximal z-component of total angular momentum!

$\downarrow \hat{J}_- = \hat{L}_- + S_-$

$$\Phi_{jj-1} = a\,Y_{\ell\ell-1}\chi_+ + b\,Y_{\ell\ell}\chi_-$$

$\downarrow$

$\vdots$

$$\boxed{\Phi_{j,-j} = Y_{\ell,-\ell}\,\chi_-}$$

$2\ell+2$ STATES: $m_j = \ell+\frac{1}{2}, \ell-\frac{1}{2}, \ell-\frac{3}{2}, \dots, -\ell-\frac{1}{2}$

- $j=\ell-\frac{1}{2}$ STATES

Φ_{j-1,m_j} DETERMINED BY REQUIRING

THEY ARE $\underline{\text{ORTHONORMAL}}$

TO ALL OTHER STATES.

2ℓ STATES: $m_j = \ell-\frac{1}{2}, \ell-\frac{3}{2}$

BETTER DO A CONCRETE EXAMPLE

$\boxed{l=1,\ s=\frac{1}{2},}$ $\quad Y_{11}\chi_\pm,\ Y_{10}\chi_\pm,\ Y_{1,-1}\chi_\pm$

$\Rightarrow j=\frac{3}{2}$ OR $j=\frac{1}{2}$

$\underline{j=\frac{3}{2}}:$ $\quad \boxed{\Phi_{\frac{3}{2},\frac{3}{2}} = Y_{11}\chi_+}$

Recall

$\frac{1}{\hbar}\hat{J}_-\Phi_{j,m_j} =$

$= \sqrt{j(j+1)-m_j(m_j-1)}\,\Phi_{j,m_j-1}$

$$\frac{1}{\hbar}\hat{J}_- = \frac{1}{\hbar}\hat{L}_- + S_-\frac{1}{\hbar}$$

$$\underbrace{\sqrt{\frac{3}{2}\frac{5}{2}-\frac{3}{2}\cdot\frac{1}{2}}}_{\sqrt{3}}\ \Phi_{\frac{3}{2},\frac{1}{2}} = \sqrt{2-0}\,Y_{10}\chi_+ + Y_{11}\sqrt{\frac{3}{4}+\frac{1}{4}}\,\chi_-)$$

$$\Rightarrow \boxed{\Phi_{\frac{3}{2},\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,Y_{10}\chi_+ + Y_{11}\chi_-\right)}$$

$$\frac{1}{\hbar}\hat{J}_- = \frac{1}{\hbar}\hat{L}_- + \frac{1}{\hbar}S_-$$

$$\underbrace{\sqrt{\frac{15}{4}-\left(-\frac{1}{4}\right)}}_{2}\ \Phi_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{\sqrt{3}}\Big(\sqrt{2}\sqrt{2}\,Y_{1,-1}\chi_+ + \sqrt{2}\,Y_{10}\chi_-$$

$$+\sqrt{2}\,Y_{10}\chi_-\Big)$$

$$\Rightarrow \boxed{\Phi_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(Y_{1,-1}\chi_+ + \sqrt{2}\,Y_{10}\chi_-\right)}$$

$$\frac{1}{\hbar}\hat{J}_- = \frac{1}{\hbar}\hat{L}_- + \frac{1}{\hbar}S_-$$

$$\underbrace{\sqrt{\frac{15}{4}-\frac{3}{4}}}_{\sqrt{3}}\ \Phi_{\frac{3}{2},-\frac{3}{2}} = \frac{1}{\sqrt{3}}\left(Y_{1,-1}\chi_- + \sqrt{2}\sqrt{2}\,Y_{1,-1}\chi_-\right)$$

$$\boxed{\Phi_{\frac{3}{2},-\frac{3}{2}} = Y_{1,-1}\chi_-}$$

$\underline{\underline{j=\frac{1}{2}}}$ $\Phi_{\frac{1}{2},\pm\frac{1}{2}}$ ARE ORTHONORMAL TO $\Phi_{\frac{3}{2},\pm\frac{1}{2}}$

$$\Rightarrow \boxed{\begin{aligned} \Phi_{\frac{1}{2},\frac{1}{2}} &= \frac{1}{\sqrt{3}}\left(Y_{10}\chi_+ - \sqrt{2}\,Y_{11}\chi_-\right) \\ \Phi_{\frac{1}{2},-\frac{1}{2}} &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\,Y_{1,-1}\chi_+ - Y_{10}\chi_-\right) \end{aligned}}$$

THIS CAN BE DONE FOR GENERAL ℓ AND INDEED FOR $s > \frac{1}{2}$: I.E. TWO ARBITRARY SPIN j_1 AND j_2 CAN BE COMBINED (ADDED) IN THIS WAY!

WEEK 12 LECT 2

BACK TO SPIN-ORBIT COUPLING

WE WANT TO FIND ENERGY SHIFTS DUE TO THE TERM

$$\hat{H}_{SO} = (\#)_{\ell,n}\frac{1}{\hbar^2}\hat{\vec{L}}\cdot\vec{S}$$

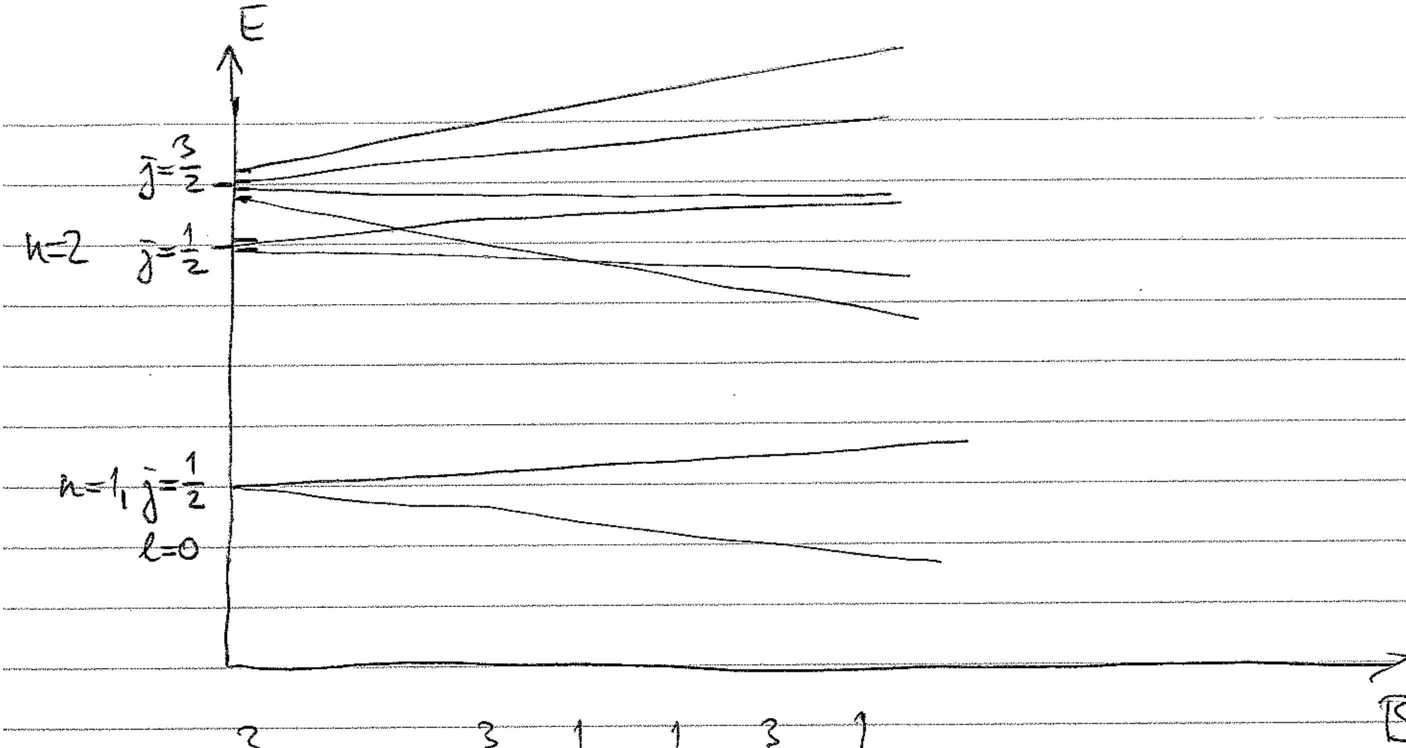
$$= (\#)_{\ell,n}\frac{1}{2\hbar^2}\left(\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \vec{S}^2\right)$$

$$\hat{H}_{SO}\left(R(r)\,\Phi_{jm_j}\right) = (\#)_{\ell n}\frac{\hbar^2}{2\hbar^2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4}\right)$$

$\underline{j = \ell + \frac{1}{2}}$: $(\#)_{\ell,n}\ \frac{1}{2}\left[\cancel{\ell^2} + \cancel{\ell}\ell + \cancel{\frac{3}{4}} - \cancel{\ell^2} - \cancel{\ell} - \cancel{\frac{3}{4}}\right] = (\#)_{\ell n}\ \frac{\ell}{2}$

$\underline{j = \ell - \frac{1}{2}}$: $(\#)_{\ell,n}\ \frac{1}{2}\left[\ell^2 - \frac{1}{4} - \ell^2 - \ell - \frac{3}{4}\right] = (\#)_{\ell,n}\ \frac{-\ell-1}{2}$

$$\hat{H}_{Zeeman} = \frac{\mu_B}{\hbar}\vec{B}\cdot(\vec{L}+2\vec{S})$$



magnetic field

$$j=\frac{3}{2} : m_j = \underbrace{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}}_{\text{4 lines}}$$

$$j=\frac{1}{2} : m_j = \underbrace{+\frac{1}{2}, -\frac{1}{2}}_{\text{2 lines}}$$

$\Big\}$ $l=1, s=\frac{1}{2}$ states

$n=2$: also $l=0 \rightarrow j=\frac{1}{2}$

$\Downarrow$

- <u>no spin S. O. effect</u>

BUT <u>ZEEMAN EFFECT</u>

Addition of A.M.

- IMPORTANT TECHNIQUE IN ATOMIC, NUCLEAR + PARTICLE PHYSICS

  - EVERYTHING IS MADE OF SPINNING PARTICLES

  - E.G. MESONS ARE MADE OF TWO SPIN $\frac{1}{2}$ QUARKS. Q? WHAT IS THE POSSIBLE TOTAL SPIN?

PARTICLE 1

$$\vec{S}_{(1)}^2 \chi_\pm^{(1)} = \frac{3\hbar^2}{4}\chi_\pm^{(1)}$$

$$S_{(1)z}\chi_\pm^{(1)} = \pm\frac{\hbar}{2}\chi_\pm^{(2)}$$

PARTICLE 2

$$\vec{S}_{(2)}\chi_\pm^{(2)} = \frac{3\hbar^2}{4}\chi_\pm^{(2)}$$

$$S_{(2)z}\chi_\pm^{(2)} = \pm\frac{\hbar}{2}\chi_\pm^{(2)}$$

TOTAL A.M. $\vec{J} = \vec{S}_{(1)} + \vec{S}_{(2)}$

$\vec{S}_{(1)}$: ONLY ACTS ON $\chi^{(1)}$

$\vec{S}_{(2)}$: ONLY ACTS ON $\chi^{(2)}$

4 POSSIBLE STATES OF COMBINED SYSTEM

o) $\chi_+^{(1)}\chi_+^{(2)} \equiv |\uparrow\uparrow\rangle$, $\chi_-^{(1)}\chi_-^{(2)} \equiv |\downarrow\downarrow\rangle$

$$J_z|\uparrow\uparrow\rangle = \left(\frac{\hbar}{2}+\frac{\hbar}{2}\right)|\uparrow\uparrow\rangle, \quad J_z|\downarrow\downarrow\rangle = \left(-\frac{\hbar}{2}-\frac{\hbar}{2}\right)|\downarrow\downarrow\rangle$$

correspond to $j=1$, $m_j=1$ OR $m_j=-1$

oo) $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$

$$J_z|\uparrow\downarrow\rangle = 0, \quad J_z|\downarrow\uparrow\rangle = 0$$

2 Possibilities for j:

$$j = s_1 + s_2 = 1 \quad \text{OR} \quad j = s_1 - s_2 = 0$$

$\underline{\underline{j=1}}$: $\Phi_{11} = |\uparrow\uparrow\rangle$

$$J_- \underbrace{|\uparrow\uparrow\rangle}_{j=1, m_j=1} = J_- \Phi_{11} = \left(S_-^{(1)} + S_-^{(2)}\right) |\uparrow\uparrow\rangle$$

$$\hbar \sqrt{j(j+1) - m(m-1)}\ \Phi_{10} = \hbar\sqrt{2}\ \Phi_{10} = \hbar |\downarrow\uparrow\rangle + \hbar |\uparrow\downarrow\rangle$$

$$\Phi_{10} = \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$\Phi_{1,-1} = |\downarrow\downarrow\rangle$$

$\underline{\underline{j=0}}$ $\quad \Phi_{00}$ ORTHOGONAL TO Φ_{10}

$$\Phi_{00} = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$$

$$\langle \phi_{10} | \phi_{00} \rangle = \frac{1}{\sqrt{2}}(a+b) = 0 \Rightarrow b = -a$$

$$\Phi_{00} = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

<u>TRIPLETT</u> $j=1$:
$m_j = 1 \quad |\uparrow\uparrow\rangle$
$m_j = 0 \quad \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$
$m_j = -1 \quad |\downarrow\downarrow\rangle$

<u>SINGLETT</u> $j=0$: $\quad m_j = 0 \quad \frac{1}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

E.G. <u>MESONS</u> CAN HAVE SPIN $\underline{\underline{j=0}}$
OR SPIN $\underline{\underline{j=1}}$

HYPERFINE SPLITTING OF HYDROGEN GROUNDSTATE ENERGY

GROUNDSTATE

$n=1,\ l=0 \Rightarrow$ NO FINESPLITTING

(FINE STRUCTURE CORRECTION $=0$ FOR $l=0$)

PROTON HAS MAGNETIC MOMENT (SPIN $=\frac{1}{2}$)

$$\vec{\mu}_p = g_p \frac{\mu_{B,p}}{\hbar} \vec{S}_p \quad \leftarrow \text{PROTON SPIN} \qquad \mu_{B,p} = \frac{e\hbar}{2m_p} \leftarrow \text{PROTON MASS}$$

$$\vec{\mu}_e = -g_s \frac{\mu_{B,e}}{\hbar} \vec{S}_e \leftarrow \text{ELECTRON SPIN} \qquad \mu_{B,e} = \frac{e\hbar}{2m_e} \leftarrow \text{ELECTRON MASS}$$

$g_s \sim 2,\ g_p \sim 5.59$

$$\hat{H}_{ep} = \underbrace{\frac{\mu_0 g_p e^2}{3\pi m_p m_e a_0^3}}_{f} \underbrace{\vec{S}_p \cdot \vec{S}_e}_{\frac{1}{2}\left(\vec{J}^2 - \vec{S}_p^2 - \vec{S}_e^2\right)}$$

$$j=1: \quad f\,\frac{1}{2}\left(\hbar^2 2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right) = f\hbar^2 \frac{1}{4}$$

$$j=0 \quad f\,\frac{1}{2}\left(\hbar^2 0 - \frac{3}{2}\hbar^2\right) = f\hbar^2\left(-\frac{3}{4}\right)$$

$$E_{j=1} - E_{j=0} = \Delta E = f\hbar^2 = h\nu$$

$\nu \downarrow$ 1450 MHz

~ 21 cm $=\lambda$ Wavelength

THE MAGNETIC MOMENT OF THE PROTON PRODUCES A MAGNETIC FIELD WHICH IS ROUGHLY SPEAKING PROPORTIONAL TO $\vec{\mu}_p$

SO $H_{ep} \propto -\vec{\mu}_e \cdot \vec{B}_{PROTON} \propto -\vec{\mu}_e \cdot \vec{\mu}_p \propto +\vec{S}_e \cdot \vec{S}_p$

- Electron Precession

SUPPOSE ELECTRON IS IN A STATE WHERE $\vec{p}, \vec{L} \sim 0$ AND IS EXPOSED TO A MAGNETIC FIELD (CONSTANT)

$$\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$$

HENCE THE TIME DEPENDENT STATE OF THE e^- IS DESCRIBED BY

$$\chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

- FIND THE TDSE AND FIND THE MOST GENERAL SOLUTION OF $\chi(t)$

- ASSUMING THAT AT $t=0$ THE e^- IS IN THE STATE:

A) $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Eigenstate of S_z

B) $\chi_+^{\prime} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Eigenstate of S_x

FIND THE EXPECTATION VALUES OF $\vec{S}$

I.E. $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$

ENERGY ONLY DUE INTERACTION OF SPIN MAGNETIC OF e^- WITH $\vec{B}$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = -\underbrace{g_s}_{\sim 2}\frac{\mu_B}{\hbar}\vec{S}$$

$$\simeq 2\frac{\mu_B}{\hbar}B S_z$$

$$= \cancel{2}\frac{\mu_B}{\cancel{\hbar}}B\frac{\cancel{\hbar}}{\cancel{2}}\sigma_z \qquad \mu_B = \frac{e\hbar}{2m_e}$$

↑ PAULI MATRIX

$$= \frac{1}{2}\underbrace{\frac{eB}{m_e}}_{\omega_c}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

ω_c .. CYCLOTRON FREQUENCY

TDSE:

$$i\hbar\frac{\partial}{\partial t}\chi = \hat{H}\chi$$

$$i\cancel{\hbar}\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = \frac{1}{2}\omega_c\cancel{\hbar}\begin{pmatrix}a\\ -b\end{pmatrix}$$

$$\Rightarrow \quad i\dot{a} = \frac{\omega_c}{2}a \quad , \dot{a} = -i\frac{\omega_c}{2}a$$

$$i\dot{b} = -\frac{\omega_c}{2}b \quad , \dot{b} = i\frac{\omega_c}{2}b$$

GENERAL SOLUTION

$$a = a_0\exp\left(-\frac{i\omega_c}{2}t\right)$$

$$b = b_0\exp\left(\frac{i\omega_c}{2}t\right)$$

$$\chi(t=0) = \begin{pmatrix}a_0\\ b_0\end{pmatrix}$$

A) $\chi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ S_z EIGENSTATE

$\Rightarrow a_0 = 1, \; b_0 = 0$

$$\chi(t) = \begin{pmatrix} e^{-i\omega_c t/2} \\ 0 \end{pmatrix}$$

$$\langle S_x \rangle = \frac{\hbar}{2} \begin{pmatrix} e^{i\omega_c t/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ 0 \end{pmatrix} = 0$$

$$\langle S_y \rangle = \frac{\hbar}{2} \begin{pmatrix} - \text{''} & - \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ 0 \end{pmatrix} = 0$$

$$\langle S_z \rangle = \frac{\hbar}{2} \begin{pmatrix} e^{i\omega_c t/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} e^{i\omega_c t/2} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ 0 \end{pmatrix} = \frac{\hbar}{2} \leftarrow \text{TIME INDEPENDENT}$$

$\langle S_x \rangle = \langle S_y \rangle = 0$ <u>NO PRECESSION</u> IN ENSEMBLE MEASUREMENT

NOTE: $[\hat{H}, S_z] = 0 \Rightarrow S_z$ CONSERVED

B) $\chi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $a_0 = b_0 = \frac{1}{\sqrt{2}}$

$$\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega_c t/2} \\ e^{i\omega_c t/2} \end{pmatrix}$$

W.F. "ROTATES" WITH HALF THE CYCLOTRON FREQUENCY

$$\langle S_x \rangle = \frac{\hbar}{2} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\omega_c t/2} & e^{-i\omega_c t/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ e^{+i\omega_c t/2} \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} e^{i\omega_c t/2} & e^{-i\omega_c t/2} \end{pmatrix} \begin{pmatrix} e^{i\omega_c t/2} \\ e^{-i\omega_c t/2} \end{pmatrix}$$

$$= \frac{\hbar}{4} \underbrace{\left(e^{i\omega_c t} + e^{-i\omega_c t} \right)}_{2\cos(\omega_c t)} = \frac{\hbar}{2} \cos(\omega_c t)$$

$$\langle S_y \rangle = \frac{\hbar}{2} \chi(t)^{\dagger} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \chi(t) = \ldots \frac{\hbar}{2} \sin(\omega_c t)$$

$$\langle S_z \rangle = \frac{\hbar}{2} \left(e^{i\omega_c t/2}, e^{-i\omega_c t/2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\omega_c t/2} \\ e^{i\omega_c t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(e^{i\omega_c t/2}, e^{-i\omega_c t/2} \right) \begin{pmatrix} e^{-i\omega_c t/2} \\ -e^{i\omega_c t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2}(1-1) = 0$$

AVERAGE S_z VALUE IS 0!

$$\chi(t) = \frac{1}{\sqrt{2}} \left[e^{-i\omega_c t/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\omega_c t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

A): $\langle \vec{S} \rangle = \left(\frac{\hbar}{2}, 0, 0 \right)$

B) $\langle \vec{S} \rangle = \frac{\hbar}{2} \left(\cos(\omega_c t), \sin(\omega_c t), 0 \right)$

PRECESSION OF EXPECTATION VALUE OF $\vec{S}$ WITH FREQUENCY ω_c !

<u>NOTE</u> $[\hat{H}, S_{x,y}] \neq 0$

$\Rightarrow$ $S_{x,y}$ NOT CONSERVED QUANTITIES

$\langle S_x \rangle$, $\langle S_y \rangle$ CAN CHANGE WITH TIME! AND THEY DO AS IN CASE B)