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PHY-413 QUANTUM MECHANICS B

EXAM 2011 SOLUTIONS

SECTION A

A1) $$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(\vec{r},t)+V(\vec{r},t)\,\Psi(\vec{r},t)$$
$$=i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \qquad \text{TDSE}$$ [2]

TISE MAYBE DERIVED FROM TDSE IF $V(\vec{r},t)=V(\vec{r})$
I.E. V IS TIME-INDEPENDENT [1]

A2) $|\Psi(x,t)|^2 dx$ = PROBABILITY TO FIND PARTICLE AT TIME t BETWEEX x AND $x+dx$
[2]

A3) FOR EVERY CLASSICAL OBSERVABLE $A(x,p_x)$ (FUNCTION OF POS. x AND MOM. p_x) THERE EXISTS A CORRESPONDING Q.M. OPERATOR $\hat{A}$ OBTAINED BY REPLACING $x\Rightarrow\hat{x}=x$, $p_x\Rightarrow\hat{p}_x=-i\hbar\frac{\partial}{\partial x}$
$$A(x,p_x)\Rightarrow\hat{A}(x,\hat{p}_x)$$
E.G. $L_z=xp_y-yp_x\Rightarrow x\hat{p}_y-y\hat{p}_x$ [3]

A4) $\hat{p}_x=-i\hbar\frac{\partial}{\partial x}$, $[x,\hat{p}_x]=i\hbar$

PROOF:
$$[x,\hat{p}_x]\psi=-i\hbar x\frac{\partial\psi}{\partial x}+i\hbar\frac{\partial}{\partial x}(x\psi)$$
$$=-i\hbar\left(\cancel{x\frac{\partial\psi}{\partial x}}-\psi-\cancel{x\frac{\partial\psi}{\partial x}}\right)=i\hbar\psi$$
TRUE $\forall\psi\Rightarrow$ OPERATOR EQN.: $[x,\hat{p}_x]=i\hbar$

A5) i) $1 = |N_0|^2 \int_{-\infty}^{+\infty} dx\, e^{-2ax^2} = \sqrt{\frac{\pi}{2a}}\, |N_0|^2$

$\Rightarrow N_0 = e^{i\varphi} \left(\frac{\pi}{2a}\right)^{-1/4}$ φ .. arbitrary phase can be set zero

$\Rightarrow \underline{\underline{N_0 = \left(\frac{\pi}{2a}\right)^{-1/4} = \left(\frac{2a}{\pi}\right)^{1/4}}}$ [2]

ii) $\langle p_x \rangle = -i\hbar \int_{-\infty}^{\infty} dx\, e^{-ax^2} \frac{\partial}{\partial x}\left(e^{-ax^2}\right) |N_0|^2$

$= -i\hbar |N_0|^2 \int_{-\infty}^{\infty} dx\, \underbrace{(-2ax)\, e^{-2ax^2}}_{\text{odd function}} \Rightarrow \underline{\underline{=0}}$ [3]

iii) $\langle p_x^2 \rangle = (-i\hbar)^2 |N_0|^2 \int_{-\infty} e^{-ax^2} \frac{\partial^2}{\partial x^2}\left(e^{-ax^2}\right)$

$= -\hbar^2 |N_0|^2 (-2a) \int_{-\infty}^{+\infty} e^{-ax^2} \frac{\partial}{\partial x}\left(e^{-ax^2} x\right) dx$

$= +2a\hbar^2 |N_0|^2 \int_{-\infty}^{+\infty} dx \left(-2ax^2 e^{-2ax^2} + e^{-2ax^2}\right)$

$= 2a\hbar^2 \left(-2\cancel{a} \frac{1}{4\cancel{a}} \sqrt{\frac{\pi}{2a}} + \sqrt{\frac{\pi}{2a}}\right) \sqrt{\frac{2a}{\pi}}$

$= 2a\hbar^2 \times \frac{1}{2} = a\hbar^2$

iv) $\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \hbar\sqrt{a}$

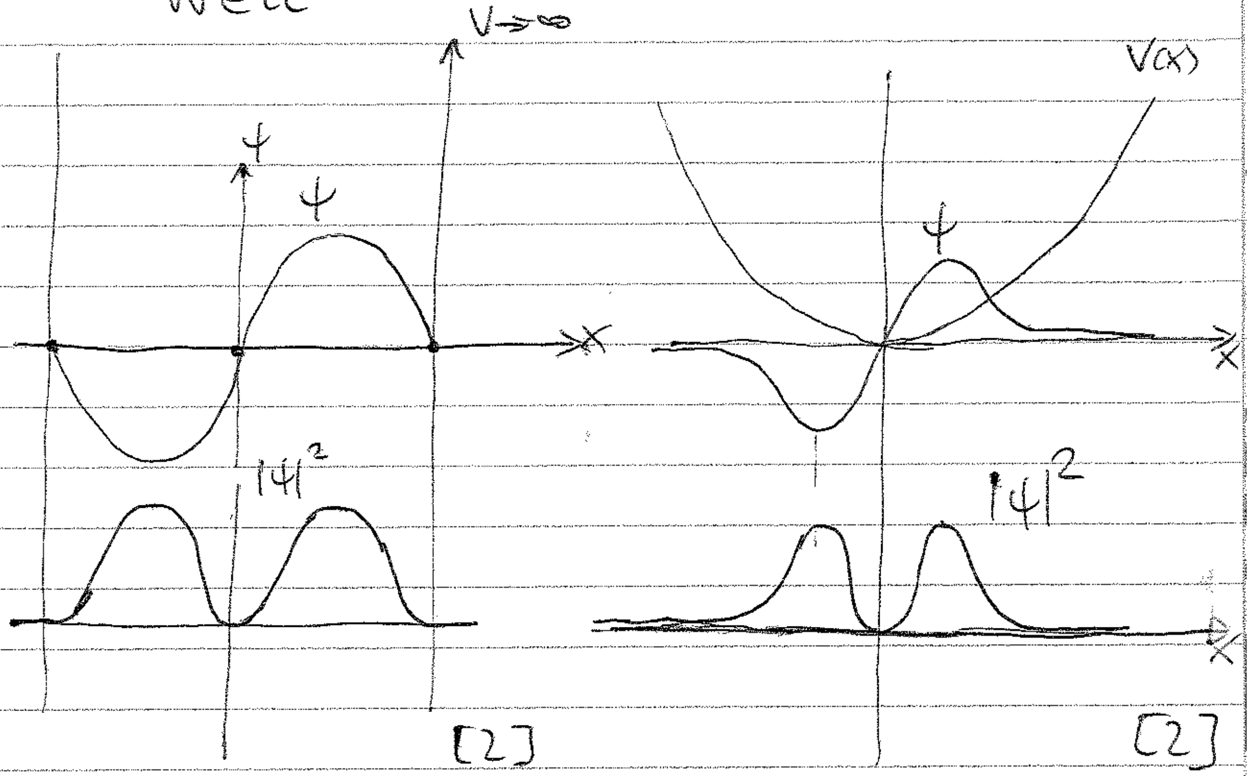
A6) $\Delta x\, \Delta p_x \geq \hbar/2$

$\Delta y\, \Delta p_y \geq \hbar/2$ OR ANY OTHER

$\Delta z\, \Delta p_z \geq \hbar/2$ EQUIVALENT FORM

③

A7) INFINITE SQUARE WELL SHO



[2] [2]

A8) i) $t>0$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\psi_0(x)\, e^{-iE_0 t/\hbar} - \frac{1}{\sqrt{2}}\psi_1(x)\, e^{-iE_1 t/\hbar}$$

[2]

ii) POSSIBLE OUTCOMES E_0 E_1

PROBABILITIES $P_0 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ $P_1 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

[2]

iii) E.G. IF MEASUREMENT YIELDS E_1
WAVEFUNCTION COLLAPSES [1]

$$\psi_{AFTER} = 1 \times \psi_1(x)\, e^{-iE_1 t/\hbar}$$

④

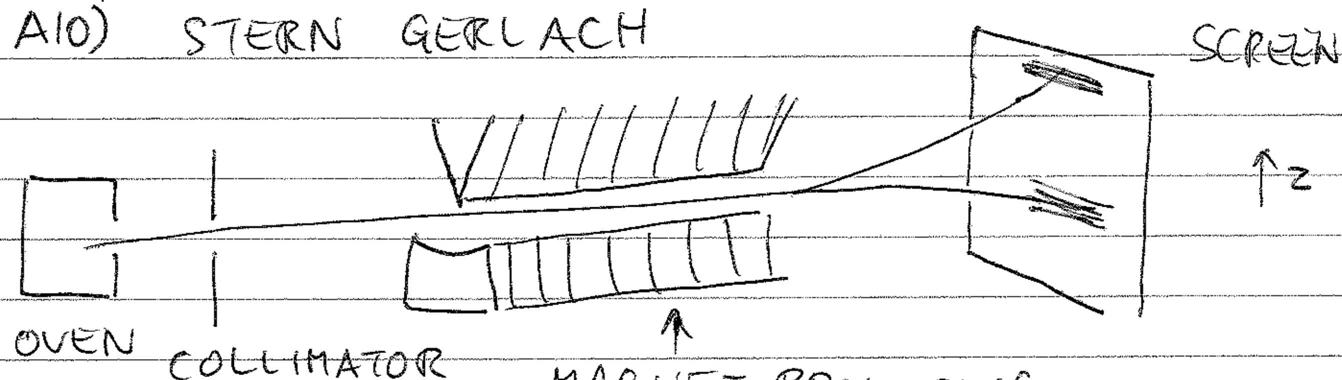
A9) $L_x = y p_z - z p_y \rightarrow \underline{\underline{\hat{L}_x = y\hat{p}_z - z\hat{p}_y}}$ [2]

E.G. USE $[x_i, \hat{p}_j] = i\hbar\delta_{ij}$

$[\hat{L}_x, y] = [y\hat{p}_z, y] - [z\hat{p}_y, y]$

$= y\underbrace{[\hat{p}_z, y]}_{=0} - z\underbrace{[\hat{p}_y, y]}_{-i\hbar} = \underline{\underline{i\hbar z}}$ [4]

A10) STERN GERLACH



$\frac{\partial B_z}{\partial z} \neq 0$

EXPERIMENT IS DESIGNED TO MEASURE z-COMPONENT OF ATOM'S MAGNETIC MOMENT $\vec{\mu}$ WHICH CAN ONLY BE DEFLECTED BY INHOMOGENEOUS MAGN. FIELD!

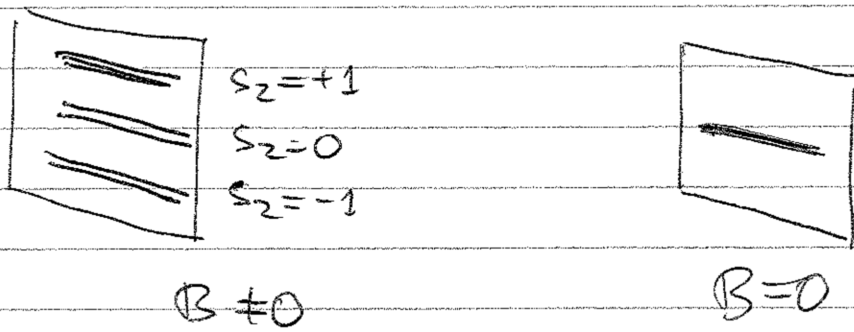
SINCE $\vec{\mu}$ IS PROP. TO (SPIN) A.M. THIS CAN BE USED TO MEASURE SPIN OF ATOMS OR ELECTRONS INSIDE ATOMS [6]

i) In Ag ALL ELECTRONS FILL SHELLS COMPLETELY ~~[illegible]~~ (TOTAL A.M. = ZERO) EXCEPT FOR ONE ($l=0$) ELECTRON (VALENCE ELECTRON) WITH SPIN = $\frac{\hbar}{2}$

(5)

THIS IS THE ONLY PART OF THE ATOM PRODUCING A MAGNETIC MOMENT (NUCLEUS CAN BE IGNORED) [3]

ii) S(spin) = 1 , MULTIPLICITY = $2S+1 = 3$



[2]

SECTION B

B1)(a) GENERAL SOLUTION OF TDSE IS

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

IF WE KNOW $\Psi(x,t=0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$

THEN $c_n = \int_{-\infty}^{+\infty} \psi_n^*(x) \Psi(x,t=0)\, dx$

(HENCE GIVEN Ψ FOR $t=0$ WE KNOW Ψ FOR $t>0$)

[4]

$\langle E \rangle = \langle \hat{H} \rangle = \int \Psi^*(x,t) \hat{H} \Psi(x,t) = ..$ USING EXPANSION THM. AND ORTHONORMALITY OF ψ_n

OR USE $P_n = |c_n|^2$ = PROBABILITY TO FIND IN A MEASUREMENT OF ENERG E_n, HENCE $\langle E \rangle$ IS WEIGHTED SUM

$$= \sum P_n E_n = \sum_{n=1}^{\infty} |c_n|^2 E_n \quad [4]$$

(b) i) $1 = |N|^2 \int_{L/4}^{L/2} dx = |N|^2 \frac{L}{4}$

$$\Rightarrow |N|^2 = \frac{4}{L} \Rightarrow N = e^{i\varphi} \frac{2}{\sqrt{L}} \xrightarrow{\varphi=0} \frac{2}{\sqrt{L}}$$

φ arbitrary

[2]

ii) PROBABILITY $P_n = |c_n|^2$ TO FIND E_n

$$c_n = \int \psi_n^* \psi(x,0)\, dx$$

$$= N \frac{\sqrt{2}}{\sqrt{L}} \int_{L/4}^{L/2} \sin\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{\sqrt{8}}{L} \frac{L}{n\pi} (-1) \cos\left(\frac{n\pi}{L} x\right) \Big|_{x=L/4}^{x=L/2}$$

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$$= \frac{\sqrt{8}}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$P_n = \frac{8}{n^2\pi^2} \underbrace{\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{n\pi}{2}\right)\right]^2}_{0 \leq \ldots \leq 4} \sim \frac{1}{n^2} \quad [4]$$

$$P_1 = \frac{8}{\pi^2}\Big[\underbrace{\cos\frac{\pi}{4}}_{=\frac{1}{\sqrt{2}}} - \underbrace{\cos\frac{\pi}{2}}_{=0}\Big]^2 = \frac{4}{\pi^2} \sim 0.41 \quad [2]$$

(c) i) $\langle [\hat{H}, \hat{O}] \rangle =$

$$= \int dx\, \psi_n^* [\hat{H}, \hat{O}] \psi_n$$

$$= \int dx \left(\psi_n^* \hat{H} \hat{O} \psi_n - \psi_n^* \hat{O} \underbrace{\hat{H} \psi_n}_{= E_n \psi_n} \right)$$

adjoint operator

$$= \int dx \left(\underbrace{(\hat{H}^\dagger \psi_n)^*}_{= (\hat{H}\psi_n)^* = (E_n \psi_n)^* = E_n \psi_n^*} \hat{O} \psi_n - E_n \psi_n^* \hat{O} \psi_n \right)$$

$\hat{H}^\dagger = \hat{H}$

$$= \int dx \left(E_n \psi_n^* \hat{O} \psi_n - E_n \psi_n^* \hat{O} \psi_n \right) = 0 \quad [4]$$

ii) $\hat{H} = \frac{1}{2m} \hat{p}_x^2 + c\, x^2 \quad , \quad \hat{O} = x \hat{p}_x$

$$[\hat{p}_x^2, x\hat{p}_x] = [\hat{p}_x^2, x]\hat{p}_x = \left(\hat{p}_x [\hat{p}_x, x] \hat{p}_x + [\hat{p}_x, x] \hat{p}_x^2\right)$$

$$= -2i\hbar\, \hat{p}_x^2$$

$$[x^2, x\hat{p}_x] = x[x^2, \hat{p}_x] = x\left(x[x, \hat{p}_x] + [x, \hat{p}_x]x\right)$$

$$= 2i\hbar\, x^2$$

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$$\Rightarrow \langle[\hat{H},\hat{O}]\rangle = 0 = -2i\hbar\underbrace{\left\langle\frac{\hat{p}_x^2}{2m}\right\rangle}_{\langle\hat{T}\rangle} + 2i\hbar\underbrace{\langle cx^2\rangle}_{\langle V\rangle} = 0$$

KINETIC ENERGY OPERATOR $\hat{T}$

$$\Rightarrow \langle\hat{T}\rangle = \langle V\rangle$$ [5]

B2) (a) $E_n = \hbar\omega_0(n+\frac{1}{2})$, $n = 0,1,2,\ldots$ [2]

(b) $[\hat{H},\hat{a}] = [\hbar\omega_0(\hat{a}^+\hat{a}+\frac{1}{2}),\hat{a}] =$

$= \hbar\omega_0[\hat{a}^+\hat{a},\hat{a}] + \cancel{[\hbar\omega_0\tfrac{1}{2},\hat{a}]}$

$= \hbar\omega_0(\hat{a}^+\cancel{[\hat{a},\hat{a}]} + \underbrace{[\hat{a}^+,\hat{a}]}_{=-1}\hat{a}) = -\hbar\omega_0\hat{a}$

$[\hat{H},\hat{a}^+] = \hbar\omega_0[\hat{a}^+\hat{a},\hat{a}^+] + \cancel{[\hbar\omega_0\tfrac{1}{2},\hat{a}^+]}$

$= \hbar\omega_0(\hat{a}^+\underbrace{[\hat{a},\hat{a}^+]}_{=1} + \cancel{[\hat{a}^+,\hat{a}^+]}\hat{a})$

$= +\hbar\omega_0\hat{a}^+$ [5]

(c) $\hat{O} = \hat{a}$ $\quad \dfrac{d\langle\hat{a}\rangle}{dt} = \dfrac{i}{\hbar}\langle[\hat{H},\hat{a}]\rangle =$

$= \dfrac{i}{\hbar}\langle-\hbar\omega_0\hat{a}\rangle = -i\omega_0\langle\hat{a}\rangle$

RECALL $\langle\hat{a}\rangle$ IS A FUNCTION OF t! (SINCE $\Psi(x,t)$ IS)

$f(t) = \langle\hat{a}\rangle$

$\Rightarrow \dot{f} = -i\omega_0 f \qquad f = Ce^{-i\omega_0 t}$

$$\underline{\underline{\langle a\rangle_t = Ce^{-i\omega_0 t}}}$$

⑨

$$\hat{O} = \hat{a}^\dagger \quad \Rightarrow \quad \frac{d\langle \hat{a}^\dagger \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{a}^\dagger] \rangle$$

$$= +i\omega_0 \langle \hat{a}^\dagger \rangle$$

$$\Rightarrow \quad \langle \hat{a}^\dagger \rangle_t = \underline{\underline{d\, e^{+i\omega_0 t}}} \qquad [5]$$

(d) $\psi = \psi_n \qquad \hat{x} = \frac{1}{\sqrt{2}\beta}(\hat{a} + \hat{a}^\dagger)$

$$\langle x \rangle = \int \psi_n^* \hat{x} \psi_n = \int \psi_n^* \frac{1}{\sqrt{2}\beta}(\hat{a}\psi_n + \hat{a}^\dagger \psi_n)$$

$$= \frac{1}{\sqrt{2}\beta} \int \psi_n^* \left(\sqrt{n}\, \psi_{n-1} + \sqrt{n+1}\, \psi_{n+1} \right)$$

$= 0$ orthonormality of ψ_n ! [2]

$$\langle x^2 \rangle = \frac{1}{2\beta^2} \int \psi_n^* \left(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} + (\hat{a}^\dagger)^2 \right) \psi_n$$

$$= \frac{\hbar}{2m\omega_0} \int \psi_n^* \Big[\cancel{\sqrt{n}\sqrt{n-1}\, \psi_{n-2}} + \hat{a}\left(\sqrt{n+1}\, \psi_{n+1}\right)$$

$$+ \hat{a}^\dagger \left(\sqrt{n}\, \psi_{n-1}\right) + \cancel{\sqrt{n+1}\sqrt{n+2}\, \psi_{n+2}} \Big]$$

orthogonality!

$$= \frac{\hbar}{2m\omega_0} \int \psi_n^* \left[(n+1)\psi_n + n\, \psi_n \right]$$

$$= \frac{\hbar}{2m\omega_0}(2n+1) = \underline{\underline{\frac{\hbar}{m\omega_0}\left(n + \frac{1}{2}\right)}} \qquad [4]$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \underline{\underline{\sqrt{\frac{\hbar}{m\omega_0}} \sqrt{n + \frac{1}{2}}}} \qquad [1]$$

(e) $\psi = \psi_0$, $\langle x^4 \rangle$ $\qquad \hat{x}^4 = \frac{1}{4\beta^4}(\hat{a} + \hat{a}^+)^4$

- THE ONLY TERMS THAT SURVIVE OUT OF 16 ARE THOSE WITH SAME # OF RAISING + LOWERING OPERATORS

- SINCE $\hat{a}\psi_0 = 0$ WE CAN DROP TERMS WITH AN $\hat{a}$ ON THE RIGHT

$$\langle \hat{x}^4 \rangle = \frac{1}{4\beta^4}\int dx\, \psi_0^* \left(\hat{a}^2 (\hat{a}^+)^2 + \hat{a}\hat{a}^+\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\hat{a}\hat{a}^+ \right)\psi_0$$

$$= \frac{1}{4\beta^4}\int dx\, \psi_0^* \left(\hat{a}^2 \sqrt{1\cdot 2}\, \psi_2 + \hat{a}\hat{a}^+\hat{a}\, \psi_1 + \hat{a}^+\hat{a}\underbrace{\hat{a}\, \psi_1}_{=\psi_0} \right)$$

$$= \frac{1}{4\beta^4}\int dx\, \psi_0^* \left(\underbrace{\sqrt{1\cdot 2}\sqrt{1\cdot 2}}_{=2}\, \psi_0 + \underbrace{\hat{a}\hat{a}^+\psi_0}_{\psi_0} \right) \quad \boxed{a\psi_0 = 0}$$

$$= \frac{3}{4\beta^4}\int dx\, \psi_0^* \psi_0 = \frac{3}{4}\frac{\hbar^2}{m^2\omega_0^2}$$ [6]

B3) a)

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

USE $[x_i, \hat{p}_j] = i\hbar\,\delta_{ij} \qquad i,j = x,y,z$

$$[\hat{L}_x, \hat{L}_y] = [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z]$$

$$= [y\hat{p}_z, z\hat{p}_x] - \cancel{[z\hat{p}_y, z\hat{p}_x]} - \cancel{[y\hat{p}_z, x\hat{p}_z]} + [z\hat{p}_y, x\hat{p}_z]$$

all operators involved commute

$$= y[\hat{p}_z, z\hat{p}_x] + \cancel{[y, z\hat{p}_x]}\hat{p}_z + z\cancel{[\hat{p}_y, x\hat{p}_z]} + [z, x\hat{p}_z]\hat{p}_y$$

$$= y\Big(\underbrace{[\hat{p}_z, z]}_{-i\hbar}\hat{p}_x + z\cancel{[\hat{p}_z, \hat{p}_x]}\Big) + \Big(\cancel{[z, x]}\hat{p}_z + x\underbrace{[z, \hat{p}_z]}_{i\hbar}\Big)\hat{p}_y$$

$$= i\hbar(x\hat{p}_y - y\hat{p}_x) = i\hbar\hat{L}_z \qquad [8]$$

b) NOTE THAT $\hat{\vec{L}}\, r^2 = 0$ HENCE $\hat{\vec{L}}\, f(r) = 0$
FOR ANY FUNCTION $f(\vec{r})$
E.G. $\hat{L}_z(x^2+y^2+z^2) = -i\hbar(x2y - y2x) = 0, \ldots$

$$\hat{L}_z \psi_1 = -Ne^{-r} L_z(x+iy)$$

$$= -Ne^{-r}(-i\hbar)\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right](x+iy)$$

$$= i\hbar N e^{-r}[x\cdot i - y]$$

$$= \hbar N e^{-r}[-x - iy] = \hbar\psi_1$$

ψ_1 EIGENVALUE : $\underline{\underline{+\hbar}}$

$$\hat{L}_z \psi_2 = Ne^{-r}\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right](x-iy)(-i\hbar)$$

$$= Ne^{-r}(-ix-y)(-i\hbar) = \hbar Ne^{-r}(-x+iy) = \underline{\underline{-\hbar\,\psi_2}}$$

ψ_2 EIGENVALUE: $\underline{\underline{-\hbar}}$

$$\hat{L}_z \psi_3 = -i\hbar Ne^{-r}\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\sqrt{2}\,z = 0 \Rightarrow \psi_3 \text{ EIGENVALUE: } \underline{\underline{0\hbar}}$$ [7]

$$\hat{L}_x + i\hat{L}_y = \hat{L}_+ = -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + iz\frac{\partial}{\partial x} - ix\frac{\partial}{\partial z}\right]$$

$$\hat{L}_+ \psi_2 = Ne^{-r}\hat{L}_+(x-iy) = -i\hbar Ne^{-r}(iz - z(-i))$$

$$= \hbar\, 2Ne^{-r}z = \hbar\sqrt{2}\,\psi_3 = \underline{\underline{c\hbar\,\psi_3}} \Rightarrow \underline{\underline{c=\sqrt{2}}}$$ [4]

(c) $\psi = N(y+z)e^{-r}$ USE EXPANSION THEOREM

$$Nye^{-r} = \frac{i}{2}(\psi_1+\psi_2), \quad Nze^{-r} = \frac{1}{\sqrt{2}}\psi_3$$

SO $\psi = \frac{i}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_3 + \frac{i}{2}\psi_2$

	$\downarrow$	$\downarrow$	$\downarrow$
EIGENVALUE	$+\hbar$	0	$-\hbar$

PROBABILITY

$P_m = |c_m|^2 \quad |c_{+1}|^2 = \left|\frac{i}{2}\right|^2 = \frac{1}{4} \quad |c_0|^2 = \frac{1}{2} \quad |c_{-1}|^2 = \frac{1}{4}$

$$\psi = c_{+1}\psi_1 + c_0\psi_2 + c_{-1}\psi_3$$ [6]

(13)

B4)

a) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ [2]

~~b)~~ $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ EIGENVALUE $+\frac{\hbar}{2}$

$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — " — $-\frac{\hbar}{2}$ [2]

$S_x \chi_+' = +\frac{\hbar}{2}\chi_+'$ $\chi_+' = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} b = a \\ a = b \end{matrix}\right\} \Rightarrow a = b$

$\chi_+' = \begin{pmatrix} a \\ a \end{pmatrix}$ NORMALISATION $(\chi_+')^\dagger \chi_+' = (a^* \; a^*)\begin{pmatrix} a \\ a \end{pmatrix}$

$= 2|a|^2 = 1$

CHOOSE $a = \frac{1}{\sqrt{2}}$

$\chi_+' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$

χ_-': $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} b \\ a \end{pmatrix} \Rightarrow \left.\begin{matrix} a = -b \\ b = -a \end{matrix}\right\} \Rightarrow b = -a$

NORMALISED $\chi_-' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+ - \chi_-)$

$\frac{1}{\sqrt{2}}(\chi_+' \pm \chi_-') = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) =$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix} \xrightarrow{+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_+$ ✓

$\xrightarrow{-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_-$ ✓ [6]

(b) $\psi = \chi_+' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $(\chi_+')^\dagger = \frac{1}{\sqrt{2}}(1 \;\; 1)$

$\langle S_y \rangle = (\chi_+')^\dagger S_y \chi_+' = \frac{1}{2}(1 \; 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{\hbar}{2}$

$= \frac{\hbar}{2}(1 \;\; 1)\begin{pmatrix} -i \\ i \end{pmatrix} = 0$

(13)

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4}\frac{1}{2}(1\ \ 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{8}(i\ \ -i)\begin{pmatrix} -i \\ i \end{pmatrix} = \frac{\hbar^2}{8}(1+1) = \frac{\hbar^2}{4}$$

$$\Rightarrow \Delta S_y = \frac{\hbar}{2}$$

$$\Delta S_y \Delta S_z = \Delta S_y \Delta S_y = \frac{\hbar^2}{4}$$ [6]

$$\Delta S_y \Delta S_z \geq \frac{1}{2}\left|\langle \underbrace{[S_y, S_z]}_{=i\hbar S_x} \rangle\right|$$

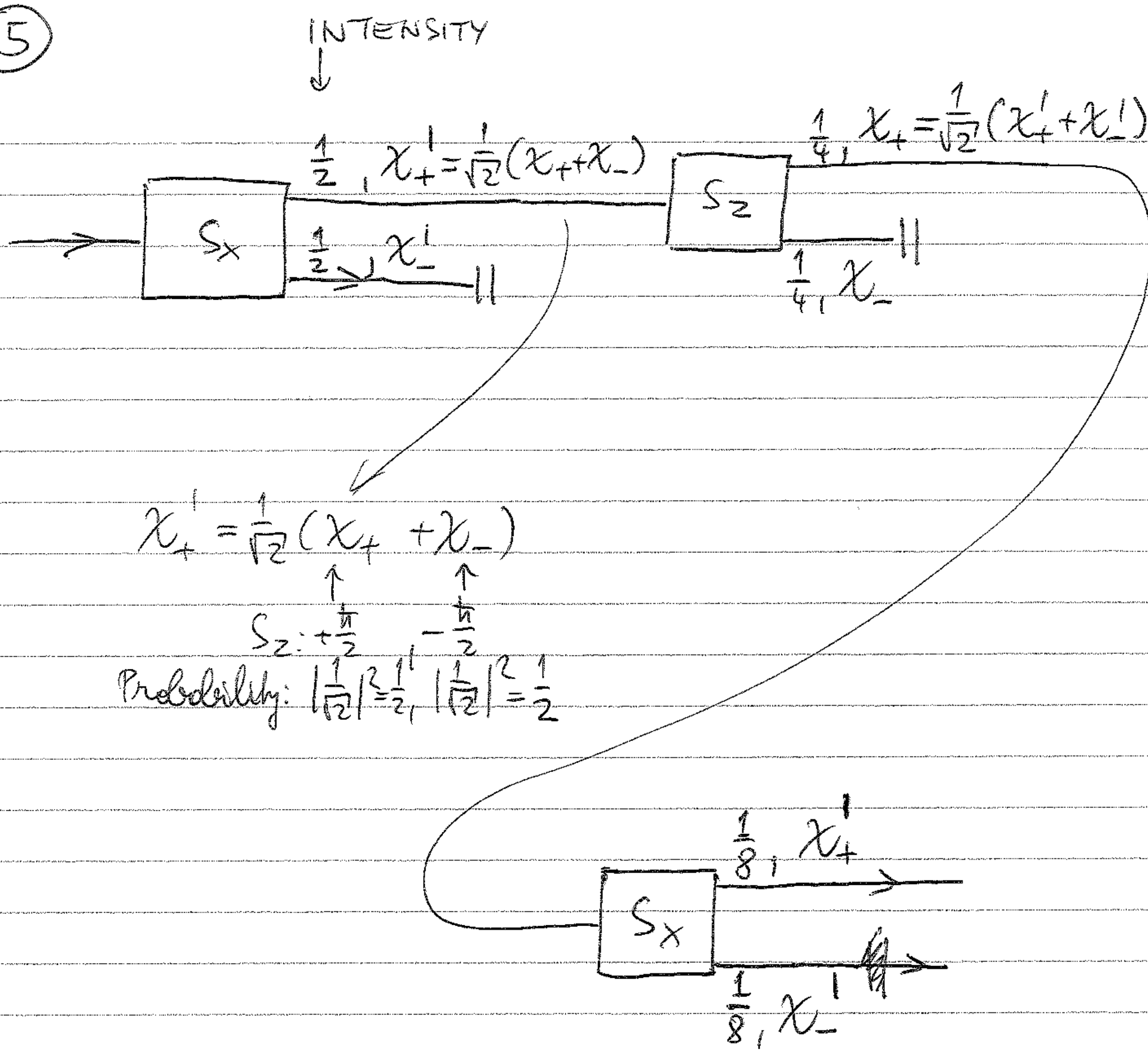
$$= \frac{\hbar}{2}\left|\langle S_x \rangle\right|$$

NOW $S_x \chi_+' = \frac{\hbar}{2}\chi_+'$

$$= \frac{\hbar}{2}\frac{\hbar}{2} = \frac{\hbar^2}{4}$$

$\Rightarrow \chi_+'$ LEADS TO MINIMUM UNCERTAINTY ALLOWED [4]

c) THE FIRST BEAM IS AN UNPOLARISED BEAM OF SPIN $\frac{1}{2}$ ATOMS, SO AFTER 1 S.G. EXPERIMENT BOTH COMPONENTS S_x $\pm\frac{\hbar}{2}$ APPEAR WITH SAME PROBABILITY $\frac{1}{2}$



$$\chi_+' = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$$

S_z: $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$

Probability: $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$, $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$