Probability current and beams of particles

Let us consider the probability density $P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$. So far we have encountered many cases where it is time independent i.e. it represents a stationary state, but what about the general case?

In general we can consider the rate of change of probability density, namely:

$$\frac{\partial P(x,t)}{\partial t}$$

where we do not expect it to be zero.

Let us consider an example of the time independent eigenstate yielding a constant probability density:

$$\Psi(x,t) = \psi_n(x)e^{-\frac{iE_n t}{\hbar}} \text{ results in } |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) = |\psi(x)|^2 \text{ with } \frac{\partial P(x,t)}{\partial t} = 0$$

This is in contrast to a linear combination of such states, for example the state:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-\frac{iE_1 t}{\hbar}} + \psi_2(x)e^{-\frac{iE_2 t}{\hbar}}\right) \text{ where, using } |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

we get $|\Psi(x,t)|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}\{\Psi_1^*(x,t)\Psi_2(x,t)\}$

or more simply $|\Psi(x,t)|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\psi_1(x)\psi_2(x)\cos(\omega_{1-2}t)$ where $\omega_{1-2} = \dfrac{E_2 - E_1}{\hbar}$

this results in the clearly non-zero $\dfrac{\partial P(x,t)}{\partial t} = 2\omega_{1-2}\,\mathrm{Im}\{\Psi_1^*(x,t)\Psi_2(x,t)\} \neq 0$

The general form of the rate of change of probability density is given by:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial(\Psi^*(x,t)\Psi(x,t))}{\partial t} = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

the quantities $\dfrac{\partial \Psi}{\partial t}$ and $\dfrac{\partial \Psi^*}{\partial t}$ can be rewritten using the TDSE:

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \text{ implies } \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi \text{ and } \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

substituting into the definition for the rate of change of probability density gives:

$$\frac{\partial P(x,t)}{\partial t} = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi = \frac{i\hbar}{2m}\frac{\partial\left\{\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right\}}{\partial x}$$

The rate of change of probability density, $\dfrac{\partial P(x,t)}{\partial t}$, is related to the probability current, *j(x,t)*, by the simple relation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}$$

so that the current itself is simply:

$$j(x,t) = -\frac{i\hbar}{2m}\left\{\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right\}$$

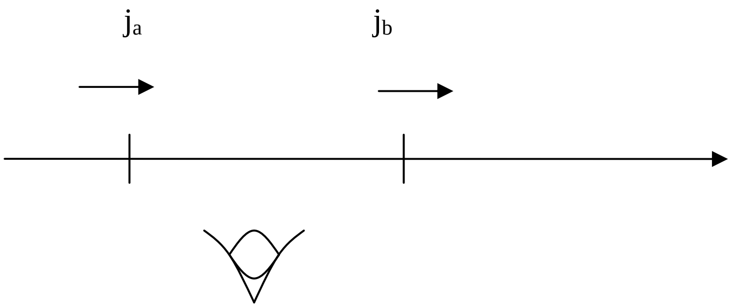
by taking into account the momentum operator, $\hat{P} = -i\hbar \frac{\partial}{\partial x}$, the current is rewritten as: $j(x,t) = \frac{1}{2m}\left\{\Psi^* \hat{P}\Psi - \Psi \hat{P}\Psi^*\right\} = \frac{1}{m}\mathrm{Re}\left[\Psi^* \hat{P}\Psi\right]$

For a De Broglie matter wave we have: $\Psi(x,t) = Ae^{\frac{i}{\hbar}(Px-Et)}$ and so the current becomes:

$$j(x,t) = \frac{P}{m}|A|^2 = v|A|^2 = \frac{\hbar k}{m}|A|^2$$

where *A* represents a number density (not a normalisation constant as the De Broglie matter wave is not normalisable).

To summarise:
The rate of change of probability density is equal to minus the current gradient



$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}$$

The current is given by:

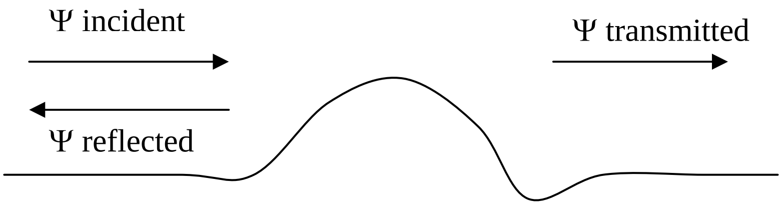
$$j(x,t) = \frac{1}{m}\mathrm{Re}\left[\Psi^* \hat{P}\Psi\right]$$

and for a De Broglie wave it is simply:

$$j(x,t) = \frac{P}{m}|A|^2 = v|A|^2 = \frac{\hbar k}{m}|A|^2$$

Reflection and Transmission – Scattering

Consider a beam incident from the left to the right towards the potential barrier shown:



Conservation of particles dictates:

$$j_{incident} = \left| j_{reflected} \right| + j_{transmitted}$$

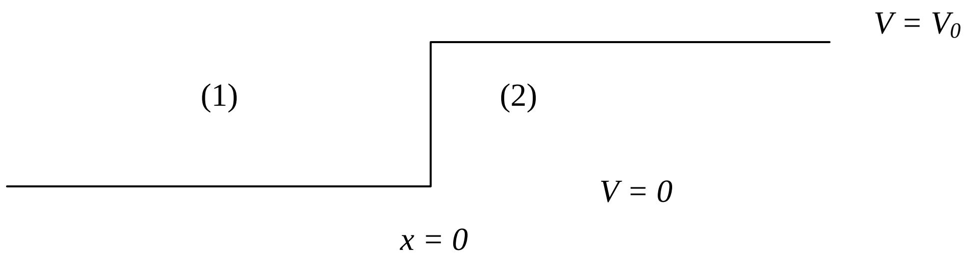
We can also define the reflection (*R*)and transmission (*T*) coefficients as:

$$R = \left| \frac{j_{ref}}{j_{inc}} \right| \qquad \text{and} \qquad T = \left| \frac{j_{trans}}{j_{inc}} \right|$$

Of course, $R + T = 1$…

The potential step

Consider a beam of particles incident from the left on the potential step shown below:



Case I: $E > V_0$

Region 1, $V = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + 0 = E\psi(x) \text{ becomes}$$

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -k^2\psi(x) \text{ where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

giving the solutions:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

which are identified as the incident and reflected waves…

Region 2, $V = V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x) \text{ becomes}$$

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi(x) = -q^2\psi(x) \text{ where } q = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

giving solutions:

$$\psi(x) = Ce^{iqx} + De^{-iqx}$$

which we reduce to: $\psi(x) = Ce^{iqx}$ since there is no source to the right of the step

We can write the probability flux as:

$$j_{inc} = |A|^2\frac{\hbar k}{m}, \qquad j_{ref} = |B|^2\frac{\hbar k}{m} \qquad \text{and} \qquad j_{trans} = |C|^2\frac{\hbar q}{m}$$

resulting in

$$R = \left|\frac{j_{ref}}{j_{inc}}\right| = \left|\frac{B}{A}\right|^2 \text{ and } T = \left|\frac{j_{trans}}{j_{inc}}\right| = \frac{q}{k}\left|\frac{C}{A}\right|^2$$

We can now derive analytical expressions for these by applying boundary conditions at $x = 0$

ψ continuous gives: $A + B = C \Rightarrow 1 + \frac{B}{A} = \frac{C}{A}$

and $\frac{d\psi}{dx}$ continuous gives: $kA - kB = qC \Rightarrow 1 - \frac{B}{A} = \frac{q}{k}\frac{C}{A}$

which together yield $\frac{C}{A} = \frac{2k}{k+q}$ and therefore $T = \frac{4qk}{(k+q)^2}$ and $R = \left(\frac{k-q}{k+q}\right)^2$

CASE II: $E < V_0$

Region 1, $V = 0$

Same as before, namely $\psi(x) = Ae^{ikx} + Be^{-ikx}$

Region 2, $V = V_0$

$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$ becomes

$\frac{d^2\psi(x)}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi(x) = \kappa^2\psi(x)$ where $\kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

giving solutions:

$\psi(x) = Ce^{-\kappa x} + De^{\kappa x}$ (decaying and rising exponentials)

which we reduce to: $\psi(x) = Ce^{-\kappa x}$ since the wavefunction must tend to zero at infinity.

Let us now consider the probability density in the two regions.

In region 1 we have:

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ and therefore $|\psi|^2 = \psi^*\psi = A^2 + B^2 + 2AB\cos 2kx$

and in region 2:

$\psi(x) = Ce^{-\kappa x}$ and therefore $|\psi|^2 = \psi^*\psi = |C|^2 e^{-2\kappa x}$

That is a stationary wave pattern in region 1 due to the interference of the incident and reflected particles and an evanescent decay in the classically forbidden region 2.