Physical Dynamics (PHY-304)

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1 Review of Newtonian Mechanics

1.1 One particle

- Lectures 1-2. Frame, velocity, acceleration, number of degrees of freedom, generalised coordinates.

- Newton's second law ($\dot{\mathbf{p}} = \mathbf{F}$), conservation of momentum.

- Angular momentum, time evolution of $\mathbf{L}$ ($\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F}$), conservation of angular momentum. Consequences of conservation of angular momentum: planar orbits, constant area velocity for the motion of two particles subject to gravitational attraction.

- Work of a force along a certain path $P(1,2)$, kinetic energy. Proof that $W[P(1,2)] = T_2 - T_1$, where $T := (m/2)\dot{\mathbf{r}}^2$. Conservative systems ($\mathbf{F} = -\vec{\nabla} V(\mathbf{r})$), potential energy.

- The work of a conservative force field is independent of the path – and viceversa, i.e. we have the theorem: A force $\mathbf{F}$ is conservative iff the work done along *any* path $P(1,2)$ only depends on the initial and final positions but not on the particular path. Energy conservation, $E(\mathbf{r}, \dot{\mathbf{r}}) := T + V$. Explicit check that $\dot{E} = 0$.

- Lecture 3. Examples of potentials. Free particle. Free falling particle, gravitational potential. Harmonic oscillator.

- One-dimensional motions. The motion with a given energy E is limited to the region where $V \leq E$. Equilibrium positions, turning points.

- The pendulum. Derivation of the equations of motion in three different ways: 1. Using Newton's equations; 2. using energy conservation; 3. using the time evolution of $\mathbf{L}$.

1.2 Many particles

- Lectures 4-5. Internal and external forces, Newton third law (also called *weak* law of action and reaction).

- Centre of mass coordinate, total momentum $\mathbf{P}$. Proof that $\dot{\mathbf{P}} = \mathbf{F}$, where $\mathbf{F}$ is the sum of all external forces (assuming the weak law of action and reaction).

- Total angular momentum $\mathbf{L}$, proof that $\dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)}$ (assuming the *strong* law of action and reaction, i.e. the interaction force between particle i and j is directed along $\mathbf{r}_{ij}$). Proof that $\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mathbf{L}'$, where $\mathbf{L}' := \sum_i \mathbf{r}'_i \times \mathbf{p}'_i$. Proof that $\dot{\mathbf{L}}' = \sum_i \mathbf{r}'_i \times \mathbf{F}_i^{(e)}$.

- Work done by the forces acting on a system of n particles. Kinetic energy, proof that $T = T_{\text{COM}} + T'$ where $T_{\text{COM}} := (1/2) \sum_i m_i \dot{\mathbf{R}}^2$ and $T' := (1/2) \sum_i m_i \dot{\mathbf{r}}'^2_i$.

- Conservative forces. Proof that if the interaction depends only on the distance, then the force is conservative. Internal and external potential energy, total energy. Energy conservation.

- Definition of rigid body, work done by the internal forces.

- Gravitational force.

2 Lagrangian Mechanics

- Lecture 6. Functionals. Calculus of variations: extremal curves of a functional, Euler-Lagrange equations. Examples: Curve of minimal length in euclidean space. Curve of minimal length on a cylinder. Independence of the choice of parametrisation. Many variables.

- Lectures 7-8. Applications to Newtonian Mechanics: Hamilton's principle of Least Action. Action, Lagrangian and Lagrange equations for a system of n particles in a potential $V(\mathbf{r}_1, \ldots, \mathbf{r}_n)$.

- Lagrangians differing by the total derivative of a function of the coordinates give rise to the same motion.

- Generalised coordinates, generalised momenta, generalised forces. Cyclic coordinates, conserved quantities. General form of the kinetic energy, $T = (1/2)a_{ij}(\mathbf{q})\dot{q}^i\dot{q}^j$.

- *(not in 2011)* Constrained systems. Constraint forces. Time-independent holonomic constraints. Variational principle for constrained systems. Equivalence of the "conditional " variational principle and the D'Alembert-Lagrange principle. Virtual displacements (tangent vectors). The work of the constraint force is zero on virtual displacements.

- General procedure for solving problems with constraints. Simple example of a constrained system: the pendulum.

- Kinetic energy and angular momentum in plane polar coordinates.

- Lecture 9. An important example: two body problem. Centre of mass coordinate and relative motion. Proof that $T' = (1/2)\mu\dot{\mathbf{r}}^2$ where $\mu := m_1m_2/(m_1 + m_2)$. Proof that $\mathbf{L}' = \mathbf{r} \times \mu\dot{\mathbf{r}}$. Decoupling of the centre of mass motion. Motion in a central field. Conservation of angular momentum, area velocity (Kepler's second law). Reduction to the equivalent one-dimensional problem, effective potential. Qualitative study of the one-dimensional effective potential. Circular orbits. Limited and unlimited orbits.

- Lectures 10-11. Derivation of the orbits for the Kepler potential. Cartesian equation of the orbits.

- Kinetic energy and angular momentum in spherical coordinates and cylindrical coordinates.

- More (simple) examples: free particle in spherical and cylindrical coordinates.

- Examples: Harmonic oscillator (pendulum), spherical pendulum. Pendulum with moving suspension point (HW). Double pendulum (HW).

- Lecture 12. Symmetries and conservation laws: Emmy Noether's theorem. Noether's conserved charges. Expression of the conserved quantity in the case of a Lagrangian exactly invariant ($\delta L = 0$) under a certain transformation: momentum/translations, angular momentum/rotations.

- Lecture 13. Expression of the conserved quantity in the case of a Lagrangian such that δL is a total time derivative, $\delta L = \frac{d}{dt}\delta F$: energy/time translation. Definition of Hamiltonian, $H := \mathbf{p} \cdot \dot{\mathbf{q}} - L$. Proof that $dH/dt = -\partial L/\partial t$. Particle in a potential with the symmetry of a helix, $V := f(z + h\phi)$.

3 Rigid Bodies

3.1 General Theory

- Lecture 14. Definition of body-fixed frame. Description of the motion in an inertial frame and in the body-fixed frame. Translational (3) and rotational (3) degrees of freedom of a rigid body. Fundamental formula of rigid kinematics (expresses the velocity of a point in the rigid body in terms of the velocity of a reference point in the body P_0 and the angular velocity ω: $\mathbf{v}_P = \mathbf{v}_{P_0} + \omega \times \overrightarrow{P_0P}$). Inertia tensor of a rigid body. Explicit expression.

- General expression of the kinetic energy of a rigid body $T = (1/2)M\dot{\mathbf{R}}^2 + (1/2)(\omega, \mathbf{I}_G\omega)$, where ω is the angular velocity in the body-fixed frame and $\mathbf{I}_G$ is the inertia tensor defined with respect to the centre of mass G, and $\mathbf{R}$ is coordinate of the centre of mass in the inertial system.

- Expression for the kinetic energy of a rigid body with a fixed point O', $T = (1/2)(\omega, \mathbf{I}'_O\omega)$.

- Lecture 15. General properties of the inertia tensor: symmetry, additivity; expression for a continuum body. Principal axes, principal moments, principal axis system of a rigid body.

- Lectures 16-17. Examples of inertia tensors: homogeneous rigid rod, sphere, cuboid. More examples in problem class and homework: ring, disk etc. Rigid bodies with an axis of symmetry.

- Rigid body rotating about a fixed axis. Moment of inertia with respect to a fixed axis $\mathbf{n}$, $I_\mathbf{n}$. Formula for $I_\mathbf{n}$, $I_\mathbf{n} = \sum_i m_i d_i^2$, where d_i is the distance of a generic point P_i in the rigid body from the axis. Kinetic energy of a rigid body rotating about a fixed axis, $T = (1/2)\omega^2 I_\mathbf{n}$.

- Parallel axis theorem (also known as Huygens-Steiner theorem).

- Solution of the physical pendulum. Small oscillations.

- Lecture 18. Angular momentum of a rigid body about a fixed point T. Proof that $\mathbf{L}_T = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{I}_G \omega$, where $\mathbf{R}$ is the centre of mass coordinate in the system with origin T and $\mathbf{I}_G$ is the inertia tensor with respect toe the centre of mass. Proof that $\mathbf{L}_G = \mathbf{I}_G \omega$, where $\mathbf{L}_G$ is the angular momentum of the rigid body with respect to the centre of mass.

- Re-expressing the kinetic energy of a rigid body as $T = (1/2)M\dot{\mathbf{R}}^2 + (1/2)\,\omega \cdot \mathbf{L}_G$.

- Re-expressing the kinetic energy of a rigid body with a fixed point O' as $T = (1/2)\,\omega \cdot \mathbf{L}_{O'}$.

3.2 Spinning Tops

- Derivation of Euler's equations.

- Study of the stability of the rotation near one of the principal axes for the asymmetric top (the "tennis racket theorem").

  Reading week

- Lectures 19-20. Solution of Euler's equations in the case of a free symmetric top ($I_1 = I_2$).

- Definition of the Euler angles (ϕ, θ, ψ). Expression of the components of the angular velocity in the body fixed-frame $(\omega_1, \omega_2, \omega_3)$ in terms of the angular velocities $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$: $\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$, $\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$, $\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$.

- Expression for the kinetic energy of a symmetric top ($I_1 = I_2$):
  $T = (I_1/2)(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + (I_3/2)(\dot{\phi} \cos\theta + \dot{\psi})^2$.

- Lecture 21. Lagrangian for the free symmetric spinning top. Conserved quantities. Precession of the top axis of symmetry about the direction of the angular momentum. Coplanarity of the top symmetry axis, ω, and the angular momentum $\mathbf{L} := L\hat{\mathbf{Z}}$. Absence of nutation ($\dot{\theta} = 0$). Expressions for the angular velocities $\dot{\phi}$, $\dot{\phi} = L/I_1$ (precession of the top symmetry axis about the direction of the angular momentum) and $\dot{\psi}$, $\dot{\psi} = L_3(I_1 - I_3)/(I_1 I_3)$ (rotation of the top about its symmetry axis), and equation for the inclination θ of the top symmetry axis, $E = \frac{L^2}{2}\left(\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3}\right)$.

- Lectures 22-23. Motion of a frisbee subject to gravity: decoupling of the centre of mass motion. Revisiting the free spinning top, making contact with Euler's equations. $\Omega = \dot{\psi}$ with $\Omega = \omega_3^0 \, (I_1 - I_3)/I_1$. Study of the rotational motion. Feynman's plate: Is the spinning about the symmetry axis faster than the wobbling? Proof that $\dot{\phi} = -2\dot{\psi}/\cos\theta \sim -2\dot{\psi}$ if the angle is slight.

- Lagrangian for the symmetric top with a fixed point in a gravity field (Lagrange's top). Study of the case $L_Z \neq L_3$. One-dimensional effective potential for the inclination θ of the top axis with respect to the vertical $\hat{\mathbf{Z}}$-direction. Qualitative analysis of the precession and nutation of the top symmetry axis.

- Sleeping top ($\theta = 0$), stability of the solution for the case $L_3^2 > 4\, Mgd\, I_1$, where d is the distance of the centre of mass from the fixed point.

- Lecture 24. Used for a double problem class.

4 Small Oscillations

- Lectures 25-26. Motivations. Definition of equilibrium position. One-dimensional case, conditions for the stability of the equilibrium position, Lagrangian of the small oscillations, frequency of small oscillations.

- Multi-dimensional case. Conditions for the stability of the equilibrium position, Lagrangian of the small oscillations. Secular (or characteristic) equation $\det(V - \omega^2 T) = 0$. Normal frequencies and normal modes.

- Lecture 27. Examples. Bead on a wire. Pendulum with moving suspension point. Double pendulum. Linear tri-atomic molecule.

5 Hamiltonian Mechanics

- Lectures 28-29. Legendre transformation. Hamiltonian and Hamilton's equations. Phase space.

- Poisson brackets. Time evolution of a physical observable $A = A(q, p, t)$: $\dot{A} = \{A, H\} + \partial A/\partial t$. Time evolution of the Hamiltonian, $\dot{H} = \partial H/\partial t = -\partial L/\partial t$.

- Examples. Free particle, harmonic oscillator, particle in a central potential.

- Lecture 30. Lagrangian and Hamiltonian for a charged particle in an electromagnetic field. Lorentz force. Gauge transformations, gauge-invariance of the action.

- Applications. Motion of a charged particle in a constant, uniform magnetic field. Solution to the equations of motion. Lagrangian for a particle in a constant magnetic field. $\mathbf{A} = 1/2\,(\mathbf{B} \times \mathbf{r})$. Noether charges associated to translations and rotations around the direction of the magnetic field.

- Lectures 31-32. Properties of the Poisson bracket. $\{A, BC\} = \{A, B\}C + B\{A, C\}$. Jacobi identity: $\{A, \{B, C\} + \{B, \{C, A\} + \{C, \{A, B\} = 0$. Proof that if A and B are two conserved quantities ($\{A, H\} = \{B, H\} = 0$), then $\{A, B\}$ is also conserved, i.e. $\{\{A, B\}, H\} = 0$ (Poisson's theorem).

6 Motion in a non-inertial frame

- Derivation of the equations of motion in a non-inertial frame. Inertial forces. Centrifugal acceleration, Coriolis acceleration.

- Foucault's pendulum.

A note on books

Please consult the list of suggested textbooks linked from the web-page of the course, here. The material discussed in the lectures can be found in most textbooks on classical/analytic mechanics. For section 1 (review of Newtonian mechanics) I mostly follow Goldstein. For the discussion of the dynamics of rigid bodies (section 3) I follow very closely the presentation of Landau and Lifschitz.