PHYSICAL DYNAMICS 2009

A1.

(i) $$\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

(ii) For each particle one has

$$\dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_{j \neq i} \vec{F}_{ji}$$

Summing over i and defining

$\vec{P} \equiv \sum_i \vec{p}_i$, $\quad \vec{F} \equiv \sum_i \vec{F}_i^{(e)}$, we get

$$\dot{\vec{P}} = \vec{F} + \underbrace{\sum_{i,j} \vec{F}_{ji}}_{= \vec{0} \text{ since } \vec{F}_{ji} = -\vec{F}_{ij}} \Rightarrow$$

$$\boxed{\dot{\vec{P}} = \vec{F}} \qquad \text{But } \vec{P} \equiv \left(\sum_i m_i\right) \dot{\vec{R}}$$

$\Rightarrow$ the centre of mass behaves as a point particle of mass $\sum_i m_i$ subject to a total force equal to $\sum_i \vec{F}_i^{(e)}$.

(iii) Set $\quad \vec{r}_i \equiv \vec{R} + \vec{r}_i\,'$

relative position with respect to the centre of mass

Then $\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i{}'$ and

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i (\dot{\vec{R}} + \dot{\vec{r}}_i{}')^2 =$$

$$= \frac{1}{2}\sum_i m_i \left(\dot{\vec{R}}^2 + \dot{\vec{r}}_i{}'^2 + 2\,\dot{\vec{R}}\cdot\dot{\vec{r}}_i{}' \right) =$$

$$= \underbrace{\frac{1}{2}\left(\sum m_i\right)\dot{\vec{R}}^2}_{\text{CENTRE OF MASS KINETIC ENERGY}} + \underbrace{\frac{1}{2}\sum_i m_i \dot{\vec{r}}_i{}'^2}_{\text{RELATIVE KINETIC ENERGY}} + \underbrace{\dot{\vec{R}}\cdot\sum_i m_i \dot{\vec{r}}_i{}'}_{\text{THIS TERM VANISHES SINCE}}$$

$$\sum_i m_i \vec{r}_i{}' = \vec{0}$$

and hence $\sum_i m_i \dot{\vec{r}}_i{}' = \vec{0}$ as well

(Note : $\sum_i m_i \vec{r}_i{}'$ is the centre of mass coordinate in the centre of mass system, i.e. it is $\vec{0}$).

A2.

(i) We need $V = V(x,z)$ i.e. V must be y-independent. Noether symmetry: translation about the $\hat{y}$-axis

(ii) We need $V = V(r)$ with $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ i.e. V must be invariant under spatial rotations.
Noether symmetry: rotations in $\mathbb{R}^3$

(iii) We need $V = V(x^2+y^2, z)$
Noether symmetry: rotations about the $\hat{z}$-axis.

A3.

(i) $$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q})$$ is the action.

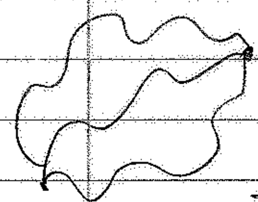
Hamilton's principle of least action:

the motion of the system is given by the function $\bar{q}(t)$ which extremises the action functional $S[q]$.

(ii) We wish to extremise $S[q]$ in the space of curves

$\{q(t) : q(t_1) = q_1, q(t_2) = q_2\}$

i.e. we hold the endpoints fixed $[\delta q(t_1) = \delta q(t_2) = 0]$



The variation δS is in this space

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} =$$

$$= \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} =$$

$$= \int_{t_1}^{t_2} dt\, \delta q \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right\} + \int_{t_1}^{t_2} d\left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

The 2nd term vanishes: $\int_{t_1}^{t_2} d\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) =$

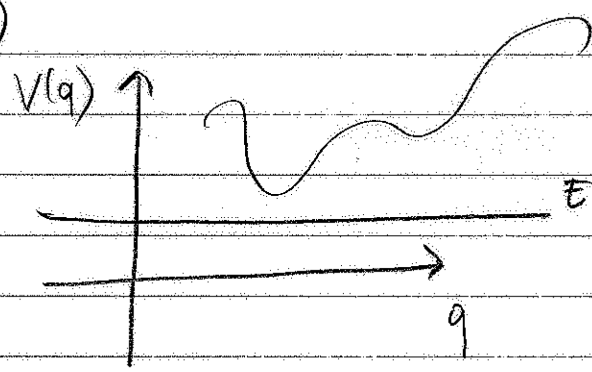
$$= \left.\frac{\partial L}{\partial \dot{q}}\right|_{t_2} \delta q(t_2) - \left.\frac{\partial L}{\partial \dot{q}}\right|_{t_1} \delta q(t_1) = 0 \text{ since } \delta q(t_{1,2}) = 0$$

As for the first term, using the arbitrariness of $\delta q(t)$ we must have

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}}$$

i.e. the Euler-Lagrange equations must hold (1 for 1 degree of freedom, as in this case).

(iii)



We have $E = T(q,\dot{q}) + V(q)$

T is a positive definite quantity $\Rightarrow$

$E - V(q) \geq 0 \quad \forall q$

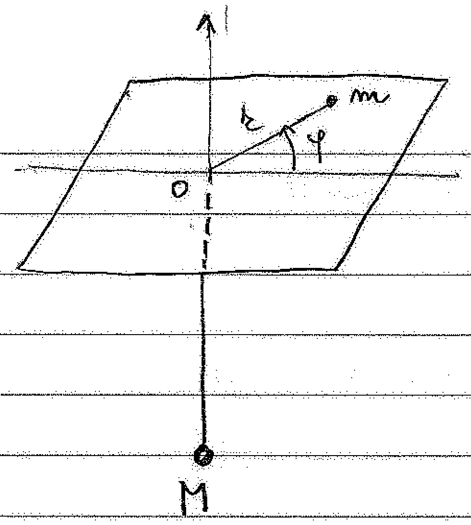
$\Rightarrow$ no motion is allowed for $E < V(q)$.

(iv) $E(q,\dot{q}) = \frac{1}{2} m \dot{q}^2 + V(q)$

$\dot{E}(q,\dot{q}) = [m\ddot{q} + V'(q)]\dot{q} = 0$

since $m\ddot{q} = -V'(q)$.

B1.



(i) There are 2 degrees of freedom, which we can parametrize using the coordinates (r, φ), where r is the distance of the mass m from the hole O, and φ is the angle as in the figure.

(ii) $T = T_m + T_M$ where

$$T_m = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2)$$ and

$$T_M = \frac{1}{2} M \dot{r}^2$$

Gravity acts on M, and $V = -Mg(l-r)$

or, up to a constant, $V = +Mgr$ $\Rightarrow$

$$\boxed{L = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - Mgr}$$

The conjugated momenta: $p_r = (m+M)\dot{r}$

$p_\varphi = m r^2 \dot{\varphi}$

Euler-Lagrange equations:

$$\boxed{\begin{cases} (m+M)\ddot{r} = m r \dot{\varphi}^2 - Mg \\ \frac{d}{dt}(m r^2 \dot{\varphi}) = 0 \end{cases}}$$

(iii) Energy is conserved [Noether symmetry: time translations]

$$E = T + V = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 + Mgr$$

Furthermore, the angular momentum of m with respect to the position of the hole is conserved [Noether symmetry: rotations about the axis $\overline{OM}$].

This is the content of the 2^{nd} Lagrange equation

$$\frac{d}{dt}\left(m r^2 \dot{\varphi}\right) = 0$$

(iv) This solution physically corresponds to a case when the centrifugal acceleration compensates gravity. In formulae,

$$\dot{r} = 0, \quad \ddot{r} = 0 \quad \text{give} \quad \boxed{mR\dot{\varphi}^2 = Mg}$$

where R is the constant distance. The (squared) angular velocity is $\boxed{\omega^2 = \frac{M}{m}\frac{g}{R}}$

We also have

$$L_z = mR^2\omega = mR^2\sqrt{\frac{M}{m}\frac{g}{R}} \Rightarrow \boxed{L_z = \sqrt{mMR^3 g}}$$

$$E = 0 + \frac{1}{2}(MgR) + MgR \Rightarrow \boxed{E = \frac{3}{2}MgR}$$

(v) Rewrite the energy as that of an effective one-dimensional problem:

$$E = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 + Mgr =$$

$= $ use $m r^2 \dot{\varphi} = L_z \equiv l$ (= constant)

$$\Rightarrow \quad E = \frac{1}{2}(m+M)\dot{r}^2 + \frac{l^2}{2mr^2} + Mgr \quad \text{or}$$

$$\boxed{E = \frac{1}{2}(m+M)\dot{r}^2 + V_{eff}(r) \quad \text{where} \quad V_{eff}(r) = Mgr + \frac{l^2}{2mr^2}}$$

$$V'_{eff}(r) = Mg - \frac{l^2}{mr^3} = 0 \quad \text{for } l^2 = mMR^3 g$$

as we saw before, or $R = \left(\frac{l^2}{mMg}\right)^{\frac{1}{3}}$

$$V''_{eff}(r) = \frac{3l^2}{mr^4}$$

At $r = R$ we have

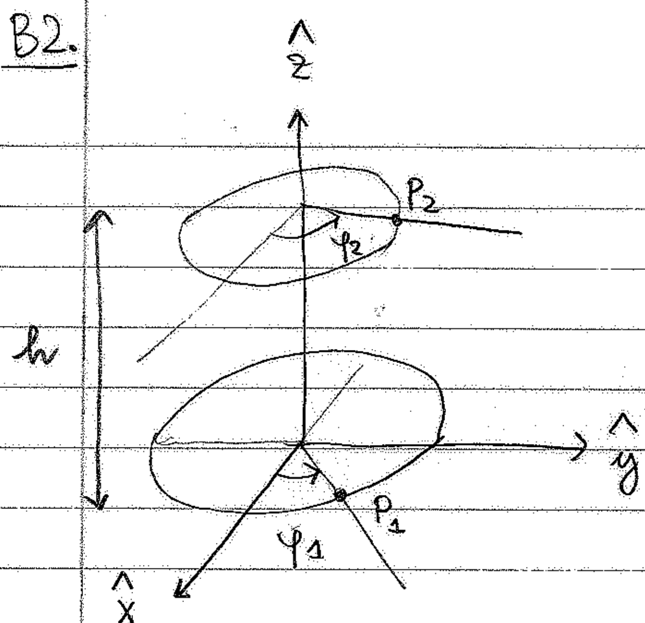
$$V''_{eff}(R) = \frac{3}{R} \cdot \frac{l^2}{mR^3} = \frac{3Mg}{R} .$$

The frequency of small oscillations is then

$$\omega^2_{s.o.} = \frac{V''(R)}{m+M} \Rightarrow \boxed{\omega^2_{s.o.} = 3\frac{g}{R}\frac{M}{m+M}}$$

<u>NOTE</u>: this is a stable equilibrium position as $V''(R) > 0$! Physically it was obvious -

B2.



(i) There are 2 degrees of freedom, parametrised by the 2 angles (φ_1, φ_2) as in the figure.

(ii) Total kinetic energy:

$$T = T_1 + T_2 = \frac{1}{2} m_1 R_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 R_2^2 \dot{\varphi}_2^2$$

The potential is $V = \frac{1}{2} k d^2$ where $d = \overline{P_1 P_2}$

Since $P_1 \equiv (R_1 \cos\varphi_1, R_1 \sin\varphi_1, 0)$

$P_2 = (R_2 \cos\varphi_2, R_2 \sin\varphi_2, h)$, we have

$$d^2 = (R_1 \cos\varphi_1 - R_2 \cos\varphi_2)^2 + (R_1 \sin\varphi_1 - R_2 \sin\varphi_2)^2 + h^2 =$$

$$= R_1^2 + R_2^2 + h^2 - R_1 R_2 (\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2) =$$

$$= -R_1 R_2 \cos(\varphi_1 - \varphi_2) + R_1^2 + R_2^2 + h^2 .$$

Hence the potential is, up to a constant,

$$V = -\frac{1}{2} k R_1 R_2 \cos(\varphi_1 - \varphi_2)$$

The Lagrangian is $L = T - V \Rightarrow$

$$\boxed{L = \frac{1}{2}\left(m_1 R_1^2 \dot{\varphi}_1^2 + m_2 R_2^2 \dot{\varphi}_2^2 \right) + \frac{1}{2} k R_1 R_2 \cos(\varphi_1 - \varphi_2)}$$

The Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_i} = \frac{\partial L}{\partial \varphi_i}$ $i = 1,2 \Rightarrow$

$$\boxed{\begin{aligned} m_1 R_1^2 \ddot{\varphi}_1 &= -\tfrac{1}{2} k R_1 R_2 \sin(\varphi_1 - \varphi_2) \\ m_2 R_2^2 \ddot{\varphi}_2 &= +\tfrac{1}{2} k R_1 R_2 \sin(\varphi_1 - \varphi_2) \end{aligned}}$$

(iii) The system is invariant under rotations about the $\hat{z}$ axis, and so is its Lagrangian. There is also invariance under time translations.

~~These associated conserved charges are~~

the projection of the angular momentum along the $\hat{z}$ axis, and the energy, respectively.

Their expressions:

L_z:
$$\boxed{\begin{aligned} L_z &= \left(m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2 \right) \cdot \hat{z} = \\ &= m_1 R_1^2 \dot{\varphi}_1 + m_2 R_2^2 \dot{\varphi}_2 \end{aligned}}$$

From summing the 2 Lagrange equations one indeed has

$$m_1 R_1^2 \ddot{\varphi}_1 + m_2 R_2^2 \ddot{\varphi}_2 = 0, \quad \text{or} \quad \dot{L}_z = 0.$$

E: The energy is $E = T + V \Rightarrow$

$$\boxed{E = \tfrac{1}{2}\left(m_1 R_1^2 \dot{\varphi}_1^2 + m_2 R_2^2 \dot{\varphi}_2^2 \right) - \tfrac{1}{2} k R_1 R_2 \cos(\varphi_1 - \varphi_2)}$$

(iv) $V = -\frac{1}{2} k R_1 R_2 \cos(\varphi_1 - \varphi_2)$

By taking the first derivatives and setting them to zero we get, at equilibrium,

$\sin(\varphi_1 - \varphi_2) = 0 \Rightarrow \varphi_1 - \varphi_2 = 0, \pi$

Notice that we cannot determine φ_1 and φ_2 separately because of the rotational symmetry of the problem.

- $\varphi_1 - \varphi_2 = 0$ is clearly a stable equilibrium position (for any value of $\varphi_1 + \varphi_2$)

Next we derive the small oscillation Lagrangian.

For convenience let us set
$$m_1 R_1^2 \equiv M_1$$
$$m_2 R_2^2 \equiv M_2$$
$$\frac{1}{2} k R_1 R_2 \equiv K$$

The original Lagrangian is

$$L = \frac{1}{2}(M_1 \dot{\varphi}_1^2 + M_2 \dot{\varphi}_2^2) + K \cos(\varphi_1 - \varphi_2)$$

Expanding about $\varphi_1 - \varphi_2 = 0$

$$V \to V_{s.o.} = \frac{K}{2}(\varphi_1 - \varphi_2)^2 = \frac{1}{2}(\varphi_1 \ \ \varphi_2)\begin{pmatrix} K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$T = \frac{1}{2}(\dot{\varphi}_1 \ \ \dot{\varphi}_2)\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}$$

The small oscillations frequencies are obtained by solving the secular equation

$$\det\left(V^{(2)} - \omega^2 T^{(2)}\right) = 0 \quad \text{where}$$

$$T^{(2)} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \qquad V^{(2)} = K\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\det\left(V^{(2)} - \omega^2 T^{(2)}\right) = \det\begin{pmatrix} K-\omega^2 M_1 & -K \\ -K & K-\omega^2 M_2 \end{pmatrix} = 0 \Rightarrow$$

$$\boxed{-\omega^2 K(M_1+M_2) + \omega^4 M_1 M_2 = 0} \Rightarrow$$

- $\omega^2 = 0$: trivial solution its presence only signals the rotational symmetry of the problem
- Non trivial solution : $\omega^2 = \dfrac{K(M_1+M_2)}{M_1 M_2}$

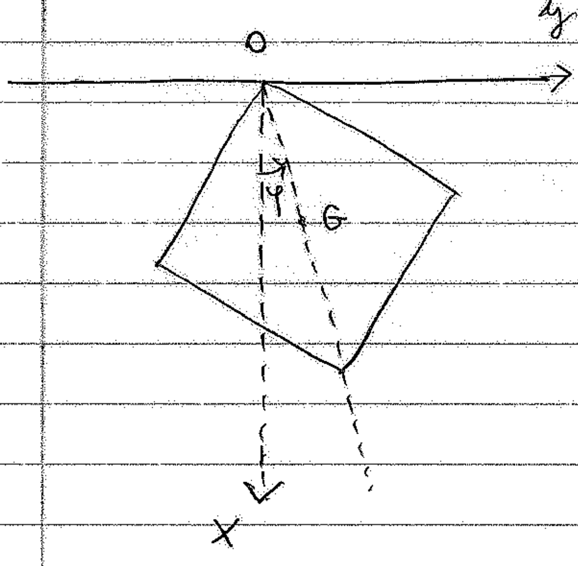
That is, using the definitions of M_1, M_2, K:

$$\boxed{\omega^2 = \frac{m_1 R_1^2 + m_2 R_2^2}{2\, m_1 m_2 R_1 R_2}\, k}$$

(V) The problem is no longer invariant under rotations about the $\hat{z}$-axis $\Rightarrow$ L_z will no longer be conserved.

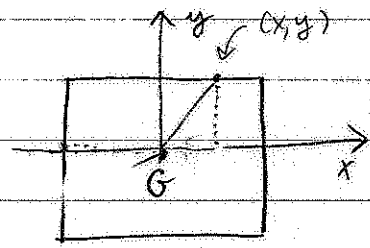
Time translation is still a symmetry $\Rightarrow$ the energy is still conserved.

B.3



(i) 1 degree of freedom parametrised by φ.

(ii)



The moment of inertia I_G with respect to an axis passing through the centre of mass G orthogonal to the square:

let $\rho = \frac{m}{\ell}$ the mass for unit length.

By symmetry reasons I_G will be equal to 4· the moment of inertia of a single rod wrt the same axis. Hence

$$I_G = 4\rho \int_{-\ell/2}^{\ell/2} dy \left(y^2 + \left(\frac{\ell}{2}\right)^2\right) = 8\rho \int_0^{\ell/2} dx \left(x^2 + \frac{\ell^2}{4}\right) =$$

$$= 8\rho\left[\frac{\ell^3}{24} + \frac{\ell^2}{4}\frac{\ell}{2}\right] = \frac{4}{3}\rho\ell^3 \qquad \text{Use } \rho = m/\ell \Rightarrow$$

$$\boxed{I_G = \frac{4}{3}m\ell^2}$$

Use next the parallel axis theorem which says that

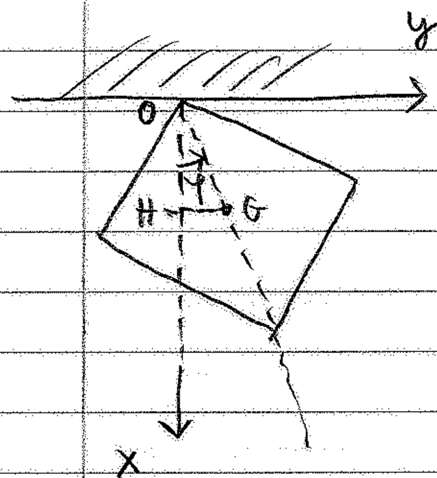
$$I_O = I_G + (\text{total mass})\cdot \overline{OG}^2 = \frac{4}{3}m\ell^2 + (4m)\cdot\left(\frac{\sqrt{2}}{2}\ell\right)^2 =$$

$$= 4m\ell^2\left(\frac{1}{3} + \frac{1}{2}\right) \Rightarrow \quad \boxed{I_O = \frac{10}{3}m\ell^2}$$

(iii) The Lagrangian for the system is

$L = T - V$ where

$$T = (T_0)_{rotational} = \frac{1}{2} I_0 \dot{\varphi}^2$$ and



$$V = -\underbrace{(4m)}_{\text{TOTAL MASS}} g\, \overline{OH} =$$

$$= -4mg \frac{\sqrt{2}}{2} \ell \cos\varphi$$

$$= -2\sqrt{2}\, mg\ell \cos\varphi$$

Using $I_0 = \frac{10}{3} m\ell^2$ we get

$$\boxed{L = \frac{5}{3} m\ell^2 \dot{\varphi}^2 + 2\sqrt{2}\, mg\ell \cos\varphi}$$

The Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi} \Rightarrow$$

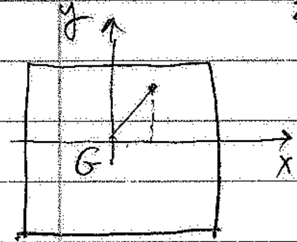
$$\frac{10}{3} m\ell^2 \ddot{\varphi} = -2\sqrt{2}\, mg\ell \sin\varphi \Rightarrow$$

$$\boxed{\ddot{\varphi} + \frac{3}{5}\sqrt{2}\frac{g}{\ell}\sin\varphi = 0}$$

(iv) $\omega^2 = \frac{3}{5}\sqrt{2}\frac{g}{\ell}$ $\qquad \omega = \left[\frac{3}{5}\sqrt{2}\frac{g}{\ell}\right]^{\frac{1}{2}}$

(v) We have to recalculate the new I_G and the new I_0. Setting $\sigma = 4m/\ell^2$ = mass per unit area,

$$\tilde{I}_G = \sigma \int_{-\frac{\ell}{2}}^{\ell/2} dx \int_{-\frac{\ell}{2}}^{\ell/2} dy\,(x^2+y^2) = 2\sigma \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dx \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dy\, y^2 =$$



$$= 2\sigma \ell \cdot \frac{2\ell^3}{24} = \frac{\sigma \ell^4}{6} = \frac{4m\ell^4}{6\ell^2} =$$

$$= \frac{2}{3} m\ell^2 \Rightarrow$$

$\tilde{I}_G = \frac{2}{3} m\ell^2$. Again using the parallel axis theorem, we have

$$\tilde{I}_0 = \tilde{I}_G + (4m)\left(\frac{\sqrt{2}}{2}\ell\right)^2 = m\ell^2\left(\frac{2}{3}+2\right) = \frac{8}{3} m\ell^2 \Rightarrow$$

$$\boxed{\tilde{I}_0 = \frac{8}{3} m\ell^2}$$

The new moment of inertia is smaller than in the case of an "empty" square (where we had $I_0 = \frac{10}{3} m\ell^2$)

The frequency of small oscillation is proportional to $\frac{1}{\sqrt{I_0}}$ (and the gravitational potential is unchanged) hence it will be LARGER (FASTER OSCILLATIONS) FOR THE "FULL" SQUARE OF (V).

In formulae, the new Lagrangian is

unchanged

$$\tilde{L} = \tilde{T} - V$$

where $\tilde{T} = \frac{1}{2}\tilde{I}_0\dot{\varphi}^2 = \frac{4}{3}ml^2\dot{\varphi}^2$

$$\tilde{L} = \frac{4}{3}ml^2\dot{\varphi}^2 + 2\sqrt{2}\,mgl\cos\varphi$$

Lagrange equations:

$$\frac{8}{3}ml^2\ddot{\varphi} = -2\sqrt{2}\,mgl\sin\varphi \Rightarrow$$

$$\boxed{\ddot{\varphi} + \frac{3\sqrt{2}}{4}\frac{g}{l}\sin\varphi}$$

$$\boxed{\tilde{\omega}^2 = \frac{3}{4}\sqrt{2}\,\frac{g}{l}}$$

The ratio new/old frequencies is

$$\frac{\tilde{\omega}^2}{\omega^2} = \frac{3\sqrt{2}}{4}\,\frac{5}{3\sqrt{2}} = \frac{5}{4} \qquad \frac{\tilde{\omega}^2}{\omega^2} = \frac{5}{4}$$

and $\boxed{\frac{\tilde{\omega}}{\omega} = \frac{\sqrt{5}}{2}}$

B4.

(i) $$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r}) \qquad \vec{p} \equiv m\dot{\vec{r}}$$

The Hamilton equations are

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad \text{where } i = 1, 2, 3.$$

(ii) $$\dot{\theta} = \frac{\partial \theta}{\partial x_i}\dot{x}_i + \frac{\partial \theta}{\partial p_i}\dot{p}_i \qquad \text{(sum over } i\text{)}$$

= using Hamilton's equations

$$= \frac{\partial \theta}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial \theta}{\partial p_i}\frac{\partial H}{\partial x_i} = \{\theta, H\}$$

where $\{A, B\}$ is the Poisson parenthesis of A with B.

(iii) $\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}$

To show that $\vec{A} \perp \vec{L}$ we calculate $\vec{A}\cdot\vec{L}$

$$\vec{A}\cdot\vec{L} = \underbrace{\vec{L}\cdot(\vec{p}\times\vec{L})}_{=0} - mk\,\underbrace{\vec{L}\cdot\frac{\vec{r}}{r}}_{=(\vec{r}\times\vec{p})\cdot\vec{r}\,=\,0}$$

$\Rightarrow \vec{A}\cdot\vec{L} = 0 \quad \Rightarrow \vec{A}$ is orthogonal to $\vec{L}$

(iv) For the case where V is a central potential, the angular momentum $\vec{L}$ is conserved (since $\dot{\vec{L}} = \vec{r}\times\vec{F} = \vec{0}$ as $\vec{F} \parallel \vec{r}$) and $\vec{r}(t)$ is always orthogonal to $\vec{L}$

$\Rightarrow$ planar orbits.

Since $\vec{A}\perp\vec{L}$, it follows that $\vec{A}$ is parallel to the plane of the orbit.

(V) $\vec{A} = \underbrace{\vec{p} \times \vec{L}} - km \frac{\vec{r}}{r}$

$$\vec{p} \times (\vec{r} \times \vec{p}) = \vec{r}\, \vec{p}^{\,2} - \vec{p}\,(\vec{r} \cdot \vec{p}) \quad \Rightarrow$$

the component i of $\vec{A}$ is

$$A_i = x_i p^2 - p_i (\vec{r} \cdot \vec{p}) - km \frac{x_i}{r}$$

$$\{A_i, H\} = \frac{\partial A_i}{\partial x_n} \frac{\partial H}{\partial p_n} - \frac{\partial A_i}{\partial p_n} \frac{\partial H}{\partial x_n}$$

We have

- $\frac{\partial A_i}{\partial x_n} = \delta_{in} p^2 - p_i p_n - km \nabla_n \left(\frac{x_i}{r} \right)$
- $\frac{\partial A_i}{\partial p_n} = 2 x_i p_n - \delta_{in} (\vec{r} \cdot \vec{p}) - p_i x_n$
- $\frac{\partial H}{\partial x_n} = \nabla_n V$
- $\frac{\partial H}{\partial p_n} = \frac{p_n}{m} \quad \Rightarrow$

$$\{A, H\} = \left[\delta_{in} p^2 - p_i p_n - km \nabla_n \left(\frac{x_i}{r} \right) \right] \frac{p_n}{m} +$$

$$- \left[2 X_i p_n - \delta_{in} (\vec{p} \cdot \vec{r}) - p_i X_n \right] \nabla_n V =$$

$$= \cancel{\frac{p_i}{m} p^2} - \cancel{p_i \frac{p^2}{m}} - k \; \vec{p} \cdot \vec{\nabla} \left(\frac{x_i}{r} \right) +$$

$$- 2 x_i \, \vec{p} \cdot \vec{\nabla} V + (\vec{p} \cdot \vec{r}) \nabla_i V + p_i \, \vec{x} \cdot \vec{\nabla} V$$

$\Rightarrow$

Next use $\vec{\nabla} V = V'(r)\, \vec{\nabla} r = V'(r) \dfrac{\vec{x}}{r}$

and $\nabla_j \left(\dfrac{x_i}{r}\right) = \dfrac{\delta_{ij}}{r} - \dfrac{x_i x_j}{r^3} = \dfrac{1}{r}\left(\delta_{ij} - \dfrac{x_i x_j}{r^2}\right)$

to get

$$\{A_i, H\} = -k\, p_j \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) +$$

$$-2x_i V'(r) \frac{\vec{p}\cdot\vec{r}}{r} + (\vec{p}\cdot\vec{r})\, V'(r) \frac{x_i}{r} + p_i \frac{r^2}{r} V'(r) =$$

$$= -k \frac{p_i}{r} + \frac{km}{r^3}(\vec{r}\cdot\vec{p})\, x_i +$$

$$- V'(r)\, x_i \frac{\vec{p}\cdot\vec{r}}{r} + V'(r)\, r\, p_i =$$

$$= \frac{p_i}{r}\left(-k + r^2 V'(r)\right) + \frac{\vec{p}\cdot\vec{r}\, x_i}{r^3}\left(k - r^2 V'(r)\right)$$

$$\Rightarrow \quad \{A_i, H\} = \left(k - r^2 V'(r)\right)\left[\frac{\vec{p}\cdot\vec{r}}{r^3} x_i - \frac{p_i}{r}\right]$$

If $V(r) = -\dfrac{k}{r}$, $V'(r) = +\dfrac{k}{r^2}$ and

$$\boxed{\{A_i, H\} = 0}$$