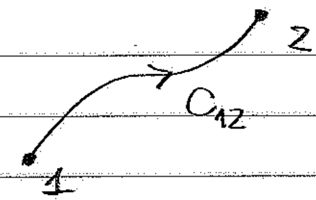
A1.

(i) $W[C_{12}] = \int_{C_{12}} d\vec{r} \cdot \vec{F}$



(ii) $\vec{F}$ is conservative if it can be written as

$$\vec{F} = -\frac{\partial}{\partial \vec{r}} V(\vec{r}) \quad \text{for some } V(\vec{r})$$

The energy is $E(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^2 + V(\vec{r})$

Its time derivative evaluated on a solution to the equations of motions is

$$\dot{E} = m \dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{\partial V}{\partial \vec{r}} \cdot \dot{\vec{r}} = \dot{\vec{r}} \cdot (m \ddot{\vec{r}} - \vec{F}) = 0$$ because of Newton's 2^{nd} law.

(iii) $W[C_{12}] = \int_{C_{12}} d\vec{r} \cdot \vec{F} = -\int_{C_{12}} d\vec{r} \cdot \frac{\partial V}{\partial \vec{r}}(\vec{r}) =$

$$= V(\vec{r}_1) - V(\vec{r}_2)$$

Hence, in the case of a conservative force the work $W[C_{12}]$ only depends on the initial and final positions, $\vec{r}_1$ and $\vec{r}_2$, but not on the specific curve C_{12} joining them.

A2.

(i) - The 3 components of $\vec{p}$; space translations
- The 3 components of $\vec{L}$; rotations
- the energy E; time translations

(ii) - The y component of $\vec{p}$; translation along the y axis, which leave $V(x,z)$ invariant.
- the energy E; time translations

(iii) - The 3 components of $\vec{L}$; rotations ($V = V(|\vec{r}|)$ is invariant under rotations about the centre of the field, O.
- the energy; time translations

(iv) - The component of $\vec{L}$ parallel to the axis of symmetry of the problem, i.e. the $\hat{z}$ axis; rotations about the $\hat{z}$ axis.
- Energy, as before.

<u>REMARK</u>: in all the above cases the kinetic term was ALWAYS invariant with respect to the symmetry transformations considered. Hence it was enough to look at the properties of the potential in each case.

A3.

(i) $L = T - V$, where the kinetic energy is $T = \frac{1}{2} m \dot{q}^2$, and $V = V(q)$

The momentum conjugated to q is

$$p \equiv \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

(ii) $H(p,q)$ is obtained from $L(q,\dot{q})$ via a Legendre transformation:

$H(p,q) = p\dot{q} - L(q,\dot{q})$, where here $\dot{q}$ should be regarded as a function of p, q implicitly defined by $p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q})$

(iii) We calculate $dH = dp\,\dot{q} + p\,d\dot{q} - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q}$

[V does not depen on t, hence $\partial H/\partial t = 0$]

Using $p \equiv \frac{\partial L}{\partial \dot{q}}$ AND the Lagrange equation $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \dot{p}$ we get

$$dH = dp\,\dot{q} + \cancel{p\,d\dot{q}} - \dot{p}\,dq - \cancel{p\,d\dot{q}} \Rightarrow \boxed{\begin{aligned} \frac{\partial H}{\partial p} &= \dot{q} \\ \frac{\partial H}{\partial q} &= -\dot{p} \end{aligned}}$$

These are the Hamilton equations.

B1.

(i) $\vec{L} = m\vec{r}\times\dot{\vec{r}}$

Taking the time derivative we obtain

$$\dot{\vec{L}} = \underbrace{m\dot{\vec{r}}\times\dot{\vec{r}}}_{=\vec{0}} + m\vec{r}\times\ddot{\vec{r}} = \vec{r}\times\vec{F}$$

But $\vec{F} = -\frac{\partial}{\partial\vec{r}}V(r) = -\frac{\partial r}{\partial\vec{r}}V'(r)$

(here $r = |\vec{r}|$). Since $\frac{\partial r}{\partial\vec{r}} = \frac{\vec{r}}{r} \equiv \hat{r}$

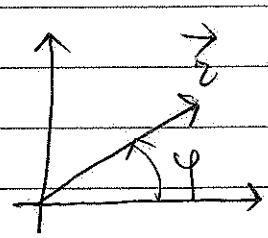
we get $\vec{r}\times\vec{F} \propto \vec{r}\times\vec{r} = \vec{0}$

Hence $\dot{\vec{L}} = \vec{0}$, $\Rightarrow \boxed{\vec{L} = \overrightarrow{\text{const}}}$

The Noether symmetries responsible for the conservation of $\vec{L}$ are space rotations about point O.

(ii) $L = T - V(r) = \frac{1}{2}m\dot{\vec{r}}^2 - V(r)$

Using $\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$, one gets



$$\boxed{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)}$$

(iii) The Lagrange equation of φ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = \frac{\partial L}{\partial \varphi}$$

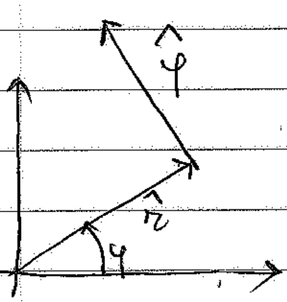
Since $\frac{\partial L}{\partial\varphi}=0$ we get $\dot p_\varphi = 0$

or $p_\varphi =$ const.

• $\boxed{p_\varphi = \frac{\partial L}{\partial\dot\varphi} = mr^2\dot\varphi}$. To see what this is

we calculate $\vec L = \vec r \times m\dot{\vec r}$

$$\vec r\times\dot{\vec r} = \vec r \times(\dot r\hat r + r\dot\varphi\hat\varphi) =$$



$$= r^2\dot\varphi(\hat r\times\hat\varphi)$$

$\hat r\times\hat\varphi$ is a unit vector

orthogonal to the plane of the motion
(i.e. pointing outside this sheet and orthogo=
=nal to it). Let us call it $\hat z$

$$\Rightarrow \vec L = \underbrace{mr^2\dot\varphi}\,\hat z \equiv p_\varphi \hat z$$

This is nothing but p_φ defined above

Hence $\boxed{\vec L = p_\varphi\hat z}$

(iv) $E(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^2 + V(r) =$

$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + V(r)$

Use $\dot{\varphi} = \frac{p_\varphi}{m r^2}$ to get

$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \frac{p_\varphi^2}{m^2 r^4} + V(r) =$

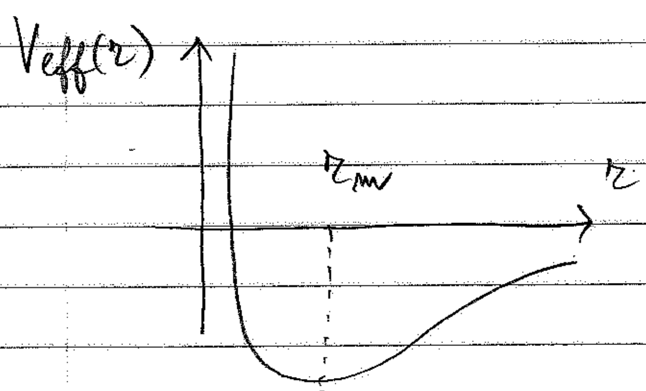
$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{p_\varphi^2}{2 m r^2} + V(r)}_{V_{eff}(r)} \Rightarrow$

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r)}$$

The problem has been reduced to a one-dimensional problem with an effective potential $\boxed{V_{eff}(r) = V(r) + \frac{p^2_\varphi}{2 m r^2}}$

(V) $V(r) = -\frac{k}{r} + \frac{p_\varphi^2}{2mr^2}$

- $r \to 0$, 2^{nd} term dominates, $V(r) \to \infty$ from above the r axis
- $r \to \infty$, 1^{st} term dominates, $V(r) \to 0$ from below the r axis $\Rightarrow$

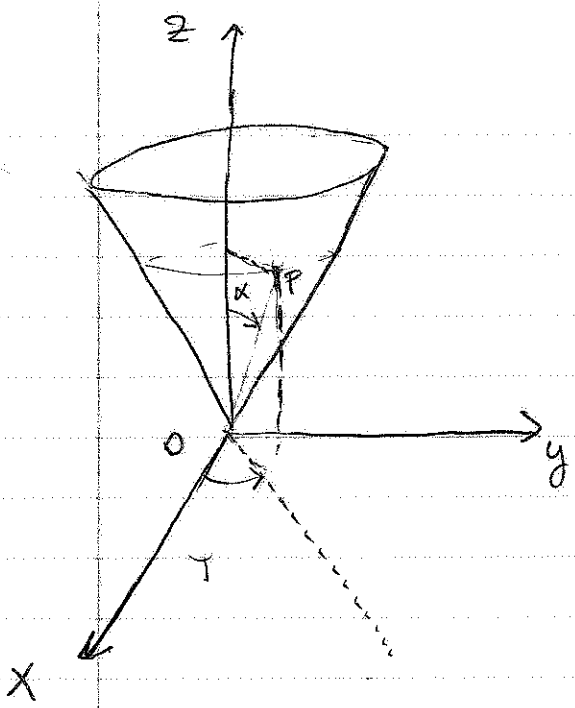


The minimum is attained for $V'_{eff}(r_m) = 0 \Rightarrow$

$$\frac{k}{r_m^2} - \frac{p_\varphi^2}{m r_m^3} = 0 \Rightarrow \boxed{r_m = \frac{p_\varphi^2}{mk}}$$

- $r(t) = r_m \ \forall t$ is a solution to the equations of motion as $V'_{eff}(r_m) = 0$.

It corresponds to an orbit with fixed radius, i.e. to a CIRCULAR ORBIT.



(i) 2 degrees of freedom

Pick spherical coordinates (r, θ, φ):

$\theta = \alpha$ is fixed, hence use

(r, φ) as generalised coordinates

where $r = \overline{OP}$

(ii)
$$\begin{cases} x = r\sin\alpha\cos\varphi \\ y = r\sin\alpha\sin\varphi \\ z = r\cos\alpha \end{cases}$$

$$\begin{cases} \dot{x} = \dot{r}\sin\alpha\cos\varphi - r\sin\alpha\sin\varphi\,\dot{\varphi} \\ \dot{y} = \dot{r}\sin\alpha\sin\varphi + r\sin\alpha\cos\varphi\,\dot{\varphi} \\ \dot{z} = \dot{r}\cos\alpha \end{cases}$$

- $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\varphi}^2)$

- $V = mgz = mgr\cos\alpha$

- $L = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\varphi}^2) - mgr\cos\alpha$

2 Euler - Lagrange equations :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r} \, , \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\alpha \, \dot{\varphi}$$

$\Rightarrow$

$$m\ddot{r} = m r \sin^2\alpha \, \dot{\varphi}^2 - mg\cos\alpha$$

$$\dot{p}_\varphi = 0 \quad \Rightarrow \quad \underline{p_\varphi \text{ IS CONSERVED}}$$

(iii) $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\dot{\vec{r}}$

$$L_z = m(x\dot{y} - y\dot{x}) =$$

$$= m \, r \sin^2\alpha \big[\cos\varphi(\cancel{\dot{r}\sin\varphi} + r\cos\varphi\,\dot{\varphi}) + - \sin\varphi(\cancel{\dot{r}\cos\varphi} - r\sin\varphi\,\dot{\varphi}) \big] =$$

$= m r^2 \sin^2\alpha \, \dot{\varphi}$, so indeed one has

$$p_\varphi = m r^2 \sin^2\alpha \, \dot{\varphi} = L_z \, .$$

- Conservation of L_z follows from Noether's theorem. The Noether symmetry associated with L_z conservation is rotation about the $\hat{z}$ axis.

(iv) $E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\varphi}^2\right) + mg\,r\cos\alpha$

Use $L_z = m r^2 \sin^2\alpha\,\dot{\varphi} = \text{constant}$

to eliminate $\dot{\varphi}$ and obtain E as a function of $r, \dot{r}$ only. We get $\left[\dot{\varphi} = \dfrac{L_z}{m r^2 \sin^2\alpha}\right]$

$$E(r,\dot{r}) = \frac{1}{2}m\dot{r}^2 + mgr\cos\alpha + \frac{1}{2}mr^2\sin^2\alpha\,\frac{L_z^2}{m^2r^4\sin^4\alpha}$$

$$= \frac{1}{2}m\dot{r}^2 + \underbrace{mgr\cos\alpha + \frac{L_z^2}{2mr^2\sin^2\alpha}}_{V_{eff}(r)}$$

We have succeeded in reducing the motion to that of a one-dimensional system with energy
$$E(r,\dot{r}) = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

where the effective potential is

$$\boxed{V_{eff}(r) = mgr\cos\alpha + \frac{L_z^2}{2mr^2\sin^2\alpha}}$$

Minimum of $V_{eff}(r)$ corresponds to a solution with $r = r_m$ = constant.

$$V'_{eff}(r) = mg\cos\alpha - \frac{L_z^2}{m r^3 \sin^2\alpha} = 0 \Rightarrow$$

$$mg\cos\alpha = \frac{L_z^2}{m r_m^3 \sin^2\alpha}$$

To determine the fine-tuned value $\dot{\varphi}_m$ of $\dot{\varphi}$ such that $r = r_m \ \forall t$, we use $L_z = m r^2 \sin^2\alpha\, \dot{\varphi}$ to get

$$\cancel{m} g \cos\alpha = \frac{\cancel{m^2} r_m^4 \sin^4\alpha\, \dot{\varphi}_m^2}{\cancel{m}\, \cancel{r_m^3}\, \cancel{\sin^2\alpha}} \Rightarrow$$

$$\boxed{\dot{\varphi}_m^2 = \frac{g\cos\alpha}{r_m \sin^2\alpha}}$$

<u>RMK</u> this value could have also been derived by noticing that for $r = r_m$, $\ddot{r} = 0$. Hence, plugging into the Lagrange' equation for r, one would get

$$0 = m r_m \sin^2\alpha\, \dot{\varphi}_m^2 - mg\cos\alpha \Rightarrow \dot{\varphi}_m^2 = \frac{g\cos\alpha}{r_m \sin^2\alpha} \text{ again.}$$

B3.

$$L = \underbrace{\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}_{T} + \underbrace{k(x\dot{y}-y\dot{x})}_{V}$$

(i) $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\vec{r}}} = \dfrac{\partial L}{\partial \vec{r}}$ gives

$$\begin{cases} \dfrac{d}{dt}(m\dot{x}-ky) = k\dot{y} \\ \dfrac{d}{dt}(m\dot{y}+kx) = -k\dot{x} \\ \dfrac{d}{dt}(m\dot{z}) = 0 \end{cases} \Rightarrow \quad \begin{aligned} &\frac{d}{dt}(m\dot{x}-2ky)=0 \\ &\frac{d}{dt}(m\dot{y}+2kx)=0 \\ &\frac{d}{dt}(m\dot{z})=0 \end{aligned}$$

$\Rightarrow$ The three quantities:

$$\begin{cases} I_x = m\dot{x}-2ky \\ I_y = m\dot{y}+2kx \\ I_z = m\dot{z} \end{cases}$$ are conserved

(ii) A translation along the z axis obviously leaves L invariant, as $z \to z+a_z$, $\dot{z}\to\dot{z}$, is a symmetry of the Kinetic energy T, and V simply does not depend on z.

The x, y translations:

again T is invariant. On the other hand

$$V \to k(x+a_x)\dot{y} - k(y+a_y)\dot{x} =$$

$$= V + a_x k\dot{y} - a_y k\dot{x} = V + \frac{d}{dt}\mathcal{F}$$

where $\mathcal{F} = k(a_x y - a_y x)$,

- Not asked in the problem, but interesting to notice:

the Noether invariants are

δa_x : $\quad \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta_x \vec{q} - \delta_x \mathcal{F} = \delta a_x [m\dot{x} - ky - ky]$

$$= \delta a_x (m\dot{x} - 2ky) = \delta a_x I_x$$

δa_y : $\quad \frac{\partial L}{\partial \dot{\vec{q}}} \delta_y \vec{q} - \delta_y \mathcal{F} = \delta a_y (m\dot{y} + ky + ky) =$

$$= \delta a_y (m\dot{y} + 2ky) = \delta a_y I_y$$

δa_z : $\quad \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta_z \vec{q} - \emptyset = \delta a_z m\dot{z} = \delta a_z I_z$

hence (I_x, I_y, I_z) are nothing but the Noether charges associated with translational invariance.

(iii) The variation of L under a rotation

$$\begin{cases} \delta x = \varepsilon y \\ \delta y = -\varepsilon x \end{cases} :$$

- T is invariant under rotations. Explicitly

$$\delta T = m(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y}) = m\varepsilon(+\dot{x}\dot{y} - \dot{y}\dot{x}) = 0$$

- The potential V:

$$\delta V = k(\delta x\, \dot{y} + x\, \delta\dot{y} - \delta y\, \dot{x} - y\, \delta\dot{x}) =$$

$$= k\varepsilon(y\dot{y} - x\dot{x} + x\dot{x} - y\dot{y}) = 0$$

The Noether charge:

$$\delta\tilde{I} = \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta\vec{q} = \varepsilon\left[(m\dot{x} - ky)y + (m\dot{y} + ky)(-x)\right] =$$

$$= \varepsilon\left[m(\dot{x}y - \dot{y}x) - k(x^2 + y^2)\right]$$

Hence the Noether invariant is

$$\boxed{\tilde{I} = m(x\dot{y} - y\dot{x}) + k(x^2 + y^2)}$$

(IV) From $I_x = m\dot{x} - 2ky$

$$I_y = m\dot{y} + 2Kx$$

one gets $I_x + iI_y = m v_\perp + 2Ki(x+iy)$

where $v_\perp = \dot{x} + i\dot{y}$

Since I_x and I_y are constant of motion it follows that

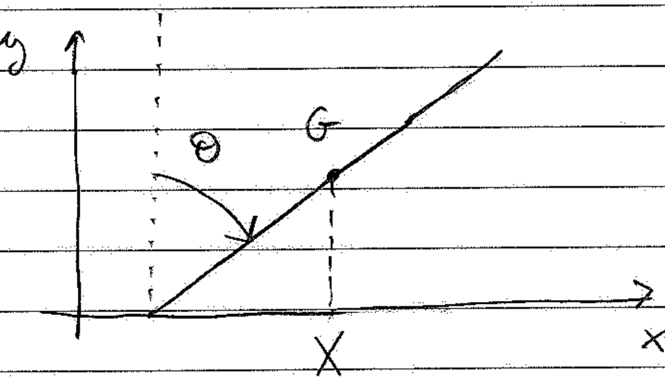
$$0 = \frac{d}{dt}(I_x + iI_y) = m\dot{v}_\perp + 2Ki v_\perp \Rightarrow$$

$$\boxed{\dot{v}_\perp = -i\frac{2K}{m} v_\perp}$$

as anticipated by the text of the problem.

B4.

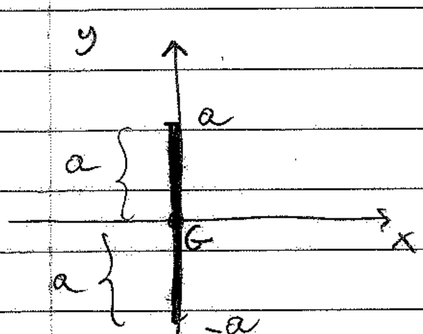
(i)



2 degrees of freedom. These could be parametrised by using the angle θ formed by the rod and the vertical and the x-coordinate X of the centre of mass G.

(ii) The moment of inertia of the rod with respect to an axis orthogonal to it and passing through G is equal to

$$I_G = \int_{-L}^{L} dy\, \rho y^2 =$$



$$= \text{using } \rho = \frac{m}{2L}$$

$$= \frac{m}{2L} \cdot \left(\frac{L^3}{3} + \frac{L^3}{3} \right) = \frac{mL^2}{3}$$

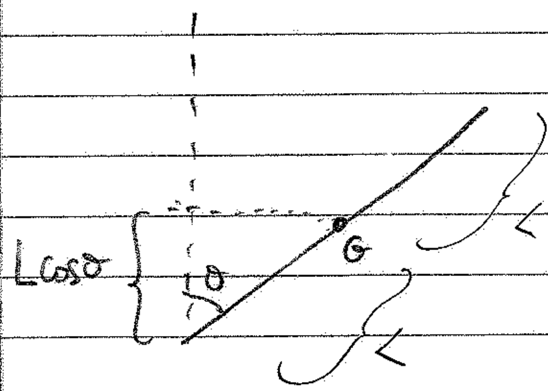
$$\boxed{I_G = \frac{mL^2}{3}}$$

(iii) The kinetic energy is

$$\boxed{T = \frac{1}{2} m(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} I_G \dot{\theta}^2}$$

where (X, Y) are the coordinates of the centre of mass G. Y is not independent; indeed $Y = L\cos\theta \Rightarrow$

$$\dot{Y} = -L\sin\theta\,\dot{\theta}$$



$$m\frac{L^2}{3}$$

Hence $T = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m L^2 \sin^2\theta\,\dot{\theta}^2 + \frac{1}{2} I_G \dot{\theta}^2$

$$\Rightarrow \boxed{T = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 \left(\sin^2\theta + \frac{1}{3}\right)}$$

• $V = mgY = mgL\cos\theta \Rightarrow$

$$\boxed{L = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 \left(\sin^2\theta + \frac{1}{3}\right) - mgL\cos\theta}$$

(iv) The coordinate X is cyclic

as $\frac{\partial L}{\partial X} = 0 \Rightarrow \dot{X} = \text{const}$

As the rod was left from rest, $\dot{x}$ will stay equal to its initial value

$\dot{X} = 0 \Rightarrow \dot{x}(t) = 0 \Rightarrow$

$$\boxed{X = \text{const}}$$