Conserved Quantities and the Evolution of Perturbations in Lemaître-Tolman-Bondi Cosmology

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Overview

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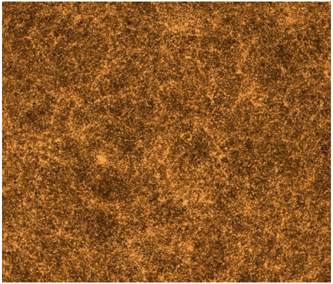
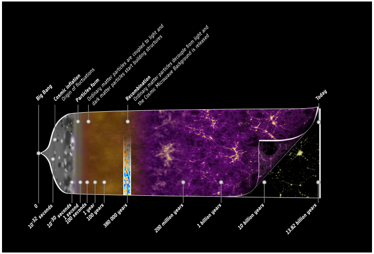


Image: SDSSIII

Why are Conserved Quantities Important?

Why are Gauge Invariant Conserved Quantities Important?

- Conserved quantities are important, allowing us to link observations at late times to the physics at earlier/all times e.g. in standard cosmology: inflation to CMB



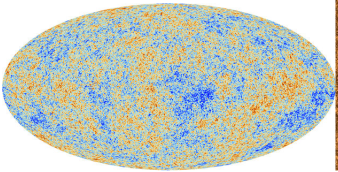
- Gauge Invariance: i.e. invariant under infinitesimal coordinate shifts $\widetilde{x^{\mu}} = x^{\mu} + \delta x^{\mu}$ to remove gauge artefacts (will come back to this)
- We construct GI conserved quantities in LTB cosmology (simplest **inhomogeneous** model)

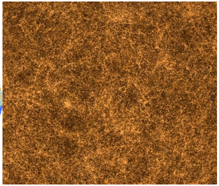
Why use flat FRW (Friedmann-Robertson-Walker) Cosmology?

Problems of Hot Big Bang Model

- Flatness: Latest Planck data - flat within 0.2%
  Early deviation from flatness rapidly grows. Why so flat now?
- Horizon: Visible horizon/Hubble radius early times << area visible today
  e.g. CMB $\approx 1\text{deg} <<$ whole sky
- Flatness: Accelerated expansion rapidly drives universe towards flat.
- Horizon: Universe inflates - superluminal expansion
  Small region in thermal equilibrium inflated beyond horizon.
- Inflation: Flattens, Causally connects.

Why use flat FRW (Friedmann-Robertson-Walker) Cosmology?





Homogeneous versus Inhomogeneous Cosmologies and Inflation

- Large Scale Structure: Beyond $\approx$ 100 Mpc $\therefore$ homogeneous.
- CMB: No temperature variation $> 10^{-5}$ K $\therefore$ homogeneous.
- Clusters and voids: small scale inhomogeneities.
- Temperature Fluctuations ($< 10^{-5}$): small scale inhomogeneities.
- Inflation driven by: simplest - scalar field, called Inflaton.
- Quantum fluctuations seed perturbations.

Flat FRW vs LTB

Flat FRW vs LTB

- FRW: Maximally symmetric spatial section - scale factor time dependent only

$$ds^2 = -dt^2 + a^2(t)dr^2 + a^2(t)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Note: LTB - Spherically symmetric spatial section - scale factors time and r coordinate dependent - **Dust dominated**

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$ [a]

[a] Bondi 1947

The Standard Model of Cosmology - Flat FRW

The Standard Model of Cosmology - Flat FRW

- Background metric:

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

- Perturbed metric:

$$ds^2 = -(1 + 2\Phi)dt^2 + 2aB_i dx^i dt + a(t)^2(\delta_{ij} + 2C_{ij})dx^i dx^j$$

with scalar, vector and tensor perturbations[a]

[a] e.g. Bardeen 1980

The Standard Model of Cosmology - Flat FRW

The Standard Model of Cosmology - Flat FRW

- Further decomposition of 3-spatial perturbations gives curvature perturbation ψ, identified with the intrinsic scalar curvature: $C_{ij} = E_{,ij} - \psi\delta_{ij}$ + vector + tensor quantities*

* On 3-spatial hypersurfaces

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Splitting quantities into background + perturbation: no longer covariant - gauge dependent; construct gauge invariant quantities
- General gauge transformations:

$$\widetilde{\delta\mathbf{T}} = \delta\mathbf{T} + \pounds_{\delta x^\mu}\bar{\mathbf{T}}$$

- Tilde denotes new coordinates

$$\widetilde{x^\mu} = x^\mu + \delta x^\mu$$

  bar denotes background.

- Key gauge transformations:

$$\begin{aligned} \widetilde{\psi_{\mathrm{FRW}}} &= \psi_{\mathrm{FRW}} + \frac{\dot{a}}{a}\delta t \\ \widetilde{\delta\rho_{\mathrm{FRW}}} &= \delta\rho_{\mathrm{FRW}} + \dot{\bar{\rho}}\delta t \end{aligned}$$

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Gauge choice: uniform density hypersurfaces, $\widetilde{\delta\rho_{\mathrm{FRW}}} = 0$

$$\delta t = -\frac{\delta\rho_{\mathrm{FRW}}}{\dot{\bar{\rho}}}$$

Get gauge invariant curvature perturbation (on constant density hypersurfaces)

$$\widetilde{\psi_{\mathrm{FRW}}}\Big|_{\widetilde{\delta\rho_{\mathrm{FRW}}}=0} = -\zeta \equiv \psi_{\mathrm{FRW}} + \frac{\dot{a}/a}{\dot{\bar{\rho}}}\delta\rho_{\mathrm{FRW}}$$

- Evolution equations for ζ from energy conservation $\nabla_\mu T^{\mu\nu} = 0$...

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Evolution equations for ζ from energy conservation $\nabla_\mu T^{\mu\nu} = 0$...
- Evolution equation for density perturbation on large scales

$$\dot{\delta\rho}_{\mathrm{FRW}} = -3H(\delta\rho_{\mathrm{FRW}} + \delta P) + 3(\bar{\rho} + \bar{P})\dot{\psi}_{\mathrm{FRW}}$$

- In the large scale limit,

$$\dot{\zeta} = -\frac{H}{\bar{\rho} + \bar{P}}\delta P_{\mathrm{nad}}$$

($\because \delta P = \delta\rho {c_s}^2 + \delta P_{\mathrm{nad}}$ and $\widetilde{\delta\rho_{\mathrm{FRW}}} = 0$ selected)

- ζ conserved for barotropic fluid in large scale limit.

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Alternatively to ζ we can construct the gauge invariant density perturbation (on flat hypersurfaces)

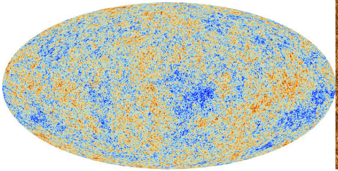
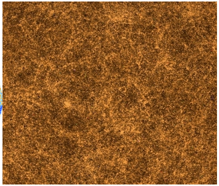
$$\delta\rho\Big|_{\widetilde{\psi_{\mathrm{FRW}}}=0} = \delta\rho_{\mathrm{FRW}} + \frac{\dot{\bar{\rho}}}{H}\psi_{\mathrm{FRW}}$$

- As stated earlier - conserved perturbed quantities allow us to easily relate early to late times (e.g. inflation to CMB)

$\because \delta\rho\Big|_{\widetilde{\psi_{\mathrm{FRW}}}=0} = -\frac{\dot{\bar{\rho}}}{H}\zeta$

$\left(\delta\rho\Big|_{\widetilde{\psi_{\mathrm{FRW}}}=0}\right.$ and ζ are GI$\left.\right)$

Why use flat FRW (Friedmann-Robertson-Walker) Cosmology?

Homogeneous versus Inhomogeneous Cosmologies and Inflation

- **But what if the universe is not homogeneous on large scales?**
- **What if the universe is inhomogeneous on some larger scale?**
- **What if the growth of structure in the universe is governed by perturbations around an inhomogeneous background cosmology?**
- **Simplest of these to investigate: Lemaître-Tolman-Bondi Cosmology**

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Perturbed LTB

- Background metric:
$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Perturbed Metric:
$$ds^2 = -(1+2\Phi)dt^2 + 2B_i d\mathcal{X}^i dt + (\delta_{ij} + 2C_{ij})d\mathcal{X}^i d\mathcal{X}^j$$
where $d\mathcal{X}^i = [X(r,t)dr, Y(r,t)d\theta, Y(r,t)\sin\theta d\phi]$ and we reserve dx^i for $[dr, d\theta, d\phi]$

-
$$X(r,t) = \frac{1}{W(r)}\frac{\partial Y(r,t)}{\partial r},$$
where $W(r)$ is an arbitrary function of r.

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Perturbed LTB

- We have performed 1+3 decomposition into time and spatial sections of metric
- In FRW we have S, V, T decomposition, here it is even more complicated and needs spherical harmonics but...[a]
- our undecomposed perturbations give simpler expressions, easing constructing conserved quantities.

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[a] e.g. Clarkson, Clifton, February 2009, Gundlach, Martin-Garcia 2000, Gerlach, Sengupta 1979

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

LTB Governing Equations

- Background Energy Conservation:

$$\dot{\rho} + \rho(H_X + 2H_Y) = 0 \,, \qquad H_X = \frac{\dot{X}}{X} \,, \qquad H_Y = \frac{\dot{Y}}{Y}$$

- Perturbed Energy Conservation:

$$\begin{aligned} \delta\dot{\rho} \quad &+ \quad (\delta\rho + \delta P)\,(H_X + 2H_Y) + \bar{\rho}' v^r \\ &+ \quad \bar{\rho}\Big(\dot{\boldsymbol{C}}_{\boldsymbol{rr}} + \dot{\boldsymbol{C}}_{\boldsymbol{\theta\theta}} + \dot{\boldsymbol{C}}_{\boldsymbol{\phi\phi}} + \partial_r v^r + \partial_\theta v^\theta + \partial_\phi v^\phi \\ &+ \quad \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right] v^r + \cot\theta v^\theta\Big) = 0 \end{aligned}$$

- Convenient to define spatial metric perturbation:

$$3\psi = \frac{1}{2}\delta g^k_k = \boldsymbol{C}_{\boldsymbol{rr}} + \boldsymbol{C}_{\boldsymbol{\theta\theta}} + \boldsymbol{C}_{\boldsymbol{\phi\phi}}$$

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

- ψ transformation behaviour:

$$3\tilde{\psi} = 3\psi + \left[\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right]\delta t + \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right]\delta r + \partial_i \delta x^i + \delta\theta \cot\theta\,,$$

- $\delta\rho$ and 3-velocity transformation behaviour:

$$\delta\tilde{\rho} = \delta\rho + \dot{\bar{\rho}}\delta t + \bar{\rho}'\delta r$$

$$\tilde{v}^i = v^i - \delta\dot{x}^i$$

- Gauge choices; uniform density (partially fixes the t coordinate):

$$\delta t\Big|_{\delta\tilde{\rho}=0} = -\frac{1}{\dot{\bar{\rho}}}\left[\delta\rho + \bar{\rho}'\delta r\right]$$

comoving (completes the gauge fixing in the spatial coordinates):

$$\delta x^i = \int v^i dt$$

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

- Note: working in gauges requiring more than one fixing condition is not unusual - c.f. synchronous gauge in standard cosmology
- Remember: perturbed flat FRW metric
$$ds^2 = -(1+2\Phi)dt^2 + 2aB_i dx^i dt + a(t)^2(\delta_{ij} + 2C_{ij})dx^i dx^j$$
- Gauge choices; $\widetilde{\Phi} = \widetilde{B_i} = 0$, means proper time for comoving observers coincides with coordinate time i.e. $d\tau = dt$
- s.t.
$$\widetilde{\psi_{\rm syn}} = \psi + H\left(\int \Phi dt - A(x^i)\right)$$
and
$$\widetilde{\delta\rho_{\rm syn}} = \delta\rho - \dot{\bar{\rho}}\left(\int \Phi dt - A(x^i)\right)$$
$A(x^i)$ is an arbitrary fn. Synchronous gauge popular in numerical simulations

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

- Gives gauge invariant **Spatial Metric Trace Perturbation (SMTP)** on comoving, uniform density hypersurfaces:

$$\begin{aligned} -\zeta_{\text{SMTP}} &= \psi + \frac{\delta\rho}{3\bar{\rho}} + \frac{1}{3}\Bigg\{ \left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r\,dt \\ &+ \partial_r \int v^r\,dt + \partial_\theta \int v^\theta\,dt + \partial_\phi \int v^\phi\,dt + \cot\theta \int v^\theta\,dt \Bigg\} \end{aligned}$$

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

- Alternatively can fix the gauge on uniform curvature hypersurfaces (i.e. $\widetilde{\psi} \equiv 0$)

$$\delta t = -\frac{1}{H_X + 2H_Y}\left[3\psi + \left(\frac{X'}{X} + 2\frac{Y'}{Y}\right)\delta r + \partial_i \delta x^i + \delta\theta \cot\theta\right]$$

- and co-moving as before to fix the δx^i terms to give

$$\begin{aligned}\delta\tilde{\rho}\Big|_{\psi=0} &= \delta\rho + \bar{\rho}\Bigg\{3\psi + \left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r dt \\ &+ \partial_r \int v^r dt + \partial_\theta \int v^\theta dt + \partial_\phi \int v^\phi dt + \cot\theta \int v^\theta dt\Bigg\}\end{aligned}$$

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

Note: it is also possible to fix the spatial gauge in a way analogous to longitudinal gauge in flat FRW i.e. set $\widetilde{B_i} = 0$

$$\delta r = -\int dt\left[\frac{\partial_r}{X^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_r}{X}\right)\right] - \int dt\left[\frac{\partial_r}{X^2}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right]$$

$$\delta\theta = -\int dt\left[\frac{\partial_\theta}{Y^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_\theta}{Y}\right)\right] - \int dt\left[\frac{\partial_\theta}{Y^2}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right]$$

$$\delta\phi = -\int dt\left[\frac{\partial_\phi}{Y^2\sin^2\theta}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \frac{B_\phi}{Y\sin\theta}\right)\right] - \int dt\left[\frac{\partial_\phi}{Y^2\sin^2\theta}\left(\frac{\delta r\bar{\rho}'}{\dot{\bar{\rho}}}\right)\right]$$

- Gauge fixing expressions far longer than constant density/constant curvature and comoving and so we did not pursue this gauge choice
- Back to constant density/constant curvature and comoving hypersurfaces...

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Constructing Gauge Invariant Quantities

- May be related to $\zeta_{\rm SMTP}$ as

$$\delta\tilde{\rho}\Big|_{\psi=0} = -3\bar{\rho}\zeta_{\rm SMTP}$$

- c.f. ζ and $\delta\tilde{\rho}\Big|_{\widetilde{\psi_{\rm FRW}}=0}$ in flat FRW

Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

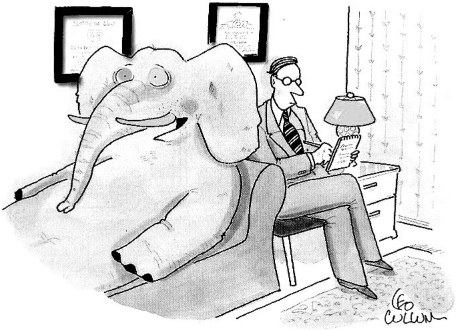
Conserved Quantities in Perturbed LTB

- ζ_{SMTP} Evolution Equation:

$$\dot{\zeta}_{\mathrm{SMTP}} = \frac{H_X + 2H_Y}{3\bar{\rho}}\delta P_{\mathrm{nad}}$$

- Valid on **all scales**.
- For barotropic fluids $\boldsymbol{\dot{\zeta}_{\mathrm{SMTP}} = 0}$

Perturbed



"I'm right there in the room, and no one even acknowledges me."

The Elephant in the Room

- Pressure perturbation around zero background?

Perturbations and Conservation in Lemaître Cosmology

Perturbed Lemaître

- Allows for background pressure (additional scale factor on the t coordinate, reduces to LTB or FRW in limits)
- Background metric:

$$ds^2 = -f^2(r,t)dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right) ,$$

- Perturbed Metric:

$$ds^2 = -f^2(r,t)(1+2\Phi)dt^2 + 2f(r,t)B_i d\mathcal{X}^i dt + (\delta_{ij} + 2C_{ij})d\mathcal{X}^i d\mathcal{X}^j$$

where $d\mathcal{X}^i = [X(r,t)dr, Y(r,t)d\theta, Y(r,t)\sin\theta d\phi]$ and we reserve dx^i for $[dr, d\theta, d\phi]$

Perturbations and Conservation in Lemaître Cosmology

Constructing Gauge Invariant Quantities

- Similar procedure to LTB: the perturbed energy conservation equation is

$$\begin{aligned}\delta\dot{\rho} &+ \left(\delta\rho+\delta P\right)\left(\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right)+(\bar{\rho}'+\bar{P}')v^r+\frac{fB_r}{X}\bar{P}'\\ &+ \left(\partial_\theta\frac{B_\theta}{Y}+\partial_\phi\frac{B_\phi}{Y\sin\theta}\right)f\bar{P}+(\bar{\rho}+\bar{P})\Big(\dot{\psi}+v^{r\prime}+\partial_\theta v^\theta+\partial_\phi v^\phi\\ &+ \left[\frac{f'}{f}+\frac{X'}{X}+2\frac{Y'}{Y}\right]v^r+\frac{B_r f'}{X}+\cot\theta v^\theta\Big)=0\,.\end{aligned}$$

- and we construct the gauge invariant quantity

$$\begin{aligned}-\zeta_{\mathrm{SMTP}} &= \psi+\frac{\delta\rho}{3(\bar{\rho}+\bar{P})}+\frac{1}{3}\left\{\left(\frac{X'}{X}+2\frac{Y'}{Y}+\frac{\bar{\rho}'}{\bar{\rho}+\bar{P}}\right)\int v^r dt\right.\\ &+ \left.\partial_r\int v^r dt+\partial_\theta\int v^\theta dt+\partial_\phi\int v^\phi dt+\cot\theta\int v^\theta dt\right\}\end{aligned}$$

Perturbations and Conservation in Lemaître Cosmology

Constructing Gauge Invariant Quantities

- The evolution equation for $\zeta_{\rm SMTP}$

$$\begin{aligned} -\dot{\zeta}_{\rm SMTP} &= \frac{\dot{\bar{\rho}}}{\left(\bar{\rho}+\bar{P}\right)^2}\delta P_{\rm nad} - \frac{\bar{P}'}{\left(\bar{\rho}+\bar{P}\right)}v^r - \frac{fB_r}{X\left(\bar{\rho}+\bar{P}\right)}\bar{P}' \\ &- \left(\partial_\theta \frac{B_\theta}{Y} + \partial_\phi \frac{B_\phi}{Y\sin\theta}\right)\frac{f\bar{P}}{\left(\bar{\rho}+\bar{P}\right)} \\ &+ \left[\partial_t\left(\frac{X'}{X}+\frac{Y'}{Y}+\frac{\bar{\rho}'}{\bar{\rho}+\bar{P}}\right)\right]\int v^r dt - \frac{f'}{f}v^r + \frac{B_r f'}{X} \end{aligned}$$

which becomes in the large scale limit (like ζ in FRW)

$$\dot{\zeta}_{\rm SMTP} = \frac{H_X + 2H_Y}{3(\bar{\rho}+\bar{P})}\delta P_{\rm nad}$$

- c.f. LTB - all scales

Perturbations and Conservation in Flat FRW Cosmology

Constructing Gauge Invariant Quantities

- The evolution equation for $\zeta_{\rm SMTP}\Big|_{\rm FRW}$ in flat FRW

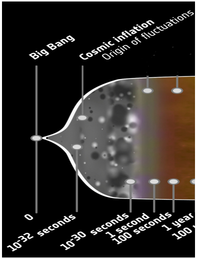
$$\dot{\zeta}_{\rm SMTP}\Big|_{\rm FRW} = \frac{H}{(\bar{\rho}+\bar{P})}\,\delta P_{\rm nad}$$

- valid on all scales and $\zeta_{\rm SMTP}\Big|_{\rm FRW}$ conserved for barotropic fluid (like LTB)($\widetilde{\delta\rho_{\rm FRW}} = 0$ and $\tilde{v} = 0$ selected).
- c.f. ζ FRW

$$\dot{\zeta} = -\frac{H}{\bar{\rho}+\bar{P}}\delta P_{\rm nad}$$

valid only in large scale limit ($\widetilde{\delta\rho_{\rm FRW}} = 0$ selected)

Further Questions of Perturbed LTB



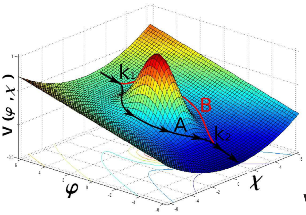
Further Questions of Perturbed LTB (Beyond scope of paper)

- At first glance LTB seems incompatible with inflation (inhomogeneous)
- Without inflation then what seeds the primordial cosmological perturbations?
- Without inflation how are problems such as relics explained?
- These arguments could apply to any inhomogeneous cosmology

Further Questions of Perturbed LTB

Further Questions of Perturbed LTB

- Some attempts made to reconcile LTB with inflation - e.g. Multistream inflation - Wang, Li 2010, more generally - Linde, Linde, Mezhlumian 1994



Wang, Li 2010

- Would allow scalar field to generate perturbations once again. Allow dilution of relic particle densities

Further Questions of Perturbed LTB

Further Questions of Perturbed LTB

- LTB not applicable at early times when background pressure important
- Other inhomogeneous cosmologies may provide an answer but...
- Potentially still needs an answer of how the universe came to be well described by an inhomogeneous model in the first place... (although Linde, Linde, Mezhlumian 1994 provide interesting argument)

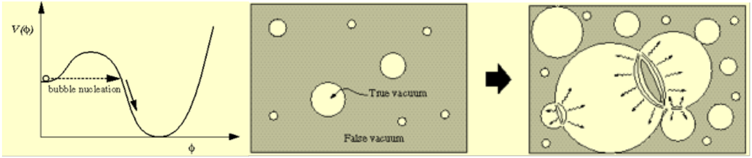
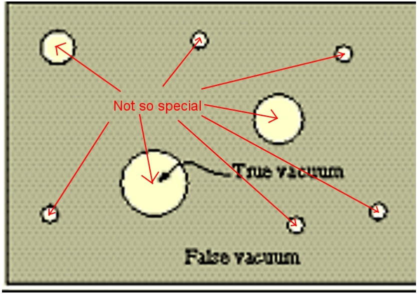


image: CTC - www.ctc.cam.ac.uk

Further Questions of Perturbed LTB

Final Word on Further Questions

- Should we discard the Copernican principle?
- If observations say we should, then yes.
- Maybe we are only throwing it away a bit - an LTB patch among many, in a larger homogeneous whole.



Perturbations and Conservation in Lemaître-Tolman-Bondi Cosmology

Conclusion and Further Research

- $$\dot{\zeta}_{\mathrm{SMTP}} = \frac{H_X + 2H_Y}{3\bar{\rho}} \delta P_{\mathrm{nad}}$$
- Research already extended to other spacetimes
  i.e. $\dot{\zeta}_{\mathrm{SMTP}}$ already extended to Lemaitre and FRW
- ζ_{SMTP} can provide a useful analytical check for numerical codes in perturbed LTB/inhomogeneous spacetimes...
  as described in 1412.3012 Bartelmann et al.

arXiv:1403.7661

Additional notes - time permitting

Additional Notes

- Comparison with 2+2 Spherical Harmonic Decomposition
- $\zeta_{\rm SMTP}$ in Clifton, Clarkson, February formalism:

$$
\begin{aligned}
-\zeta_{\rm SMTP} \quad = \quad & \frac{1}{6}\left(\frac{(1-\kappa r^2)}{a^2{}_{\parallel}} h_{rr}\mathcal{Y} + \frac{h\ \bar{\mathcal{Y}}_{\theta\theta}}{a_{\perp}{}^2 r^2} + \frac{h\ \bar{\mathcal{Y}}_{\phi\phi}}{a_{\perp}{}^2 r^2 \sin^2\theta} + 2K\mathcal{Y} + G\mathcal{Y}_{:\theta\theta} + \frac{G\mathcal{Y}_{:\phi\phi}}{\sin^2\theta}\right) \\
+ \quad & \frac{\delta\rho}{3\bar{\rho}} + \frac{1}{3}\left\{\partial_\theta \int \frac{1}{a_{\perp}{}^2 r^2}\left(\bar{\rm v}\ \bar{\mathcal{Y}}_\theta + \tilde{\rm v}\ \mathcal{Y}_\theta - h_t^{\rm axial}\bar{\mathcal{Y}}_\theta - h_t^{\rm polar}\mathcal{Y}_\theta\right) dt\right. \\
+ \quad & \partial_\phi \int \frac{1}{a_{\perp}{}^2 r^2 \sin^2\theta}\left(\bar{\rm v}\ \bar{\mathcal{Y}}_\phi + \tilde{\rm v}\ \mathcal{Y}_\phi - h_t^{\rm axial}\bar{\mathcal{Y}}_\phi - h_t^{\rm polar}\mathcal{Y}_\phi\right) dt \\
+ \quad & \cot\theta \int \frac{1}{a_{\perp}{}^2 r^2}\left(\bar{\rm v}\ \bar{\mathcal{Y}}_\theta + \tilde{\rm v}\ \mathcal{Y}_\theta - h_t^{\rm axial}\bar{\mathcal{Y}}_\theta - h_t^{\rm polar}\mathcal{Y}_\theta\right) dt \\
- \quad & \partial_r \int \frac{\mathcal{Y}(1-\kappa r^2)}{a_{\parallel}{}^2}\left(\frac{1}{2}h_{tr} + \frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right) dt \\
- \quad & \left[\left(\frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\right)^{\dagger} + 2\frac{(a_{\perp} r)^{\dagger} a_{\parallel}}{a_{\perp} r\sqrt{(1-\kappa r^2)}}\right. \\
+ \quad & \left.\frac{\bar{\rho}^{\dagger} a_{\parallel}}{\bar{\rho}\sqrt{(1-\kappa r^2)}}\right] \int \frac{(1-\kappa r^2)}{a^2{}_{\parallel}}\mathcal{Y}\left(\frac{1}{2}h_{tr} + \frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right) dt \Bigg\},
\end{aligned}
$$