Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-413 Quantum Mechanics B

Time Allowed: 2 hours 30 minutes

Date: 10^{th} June, 2011

Time: 10:00 - 12:30

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work and mid-term test comprise 20% of the final mark.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A.Brandhuber, Dr. T.Kreouzis

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SECTION A. Attempt answers to all questions.

A1 Write down the Time-Dependent Schrödinger Equation (TDSE) for a particle of mass m in 3-D in a potential $V(\vec{r},t)$. State under what conditions for the potential $V(\vec{r},t)$ the Time Independent Schrödinger Equation may be derived from the Time-Dependent one. **[3]**

A2 State the Born interpretation for a normalised single particle wave function $\Psi(x,t)$ in one dimension. **[2]**

A3 What is meant by the Operator Postulate in Quantum Mechanics? **[3]**

A4 Write down the commutation relation of the position operator $\hat{x}$ and the momentum operator $\hat{p}_x$. Using the representation for the momentum operator, provide a proof. **[3]**

A5 For the ground state wave function of the simple harmonic oscillator

$$\Psi_0(x,0) = N_0\, e^{-ax^2}$$

calculate

i) the normalisation constant N_0, **[2]**

ii) the expectation value $\langle \hat{p}_x \rangle$, **[3]**

iii) the expectation value $\langle \hat{p}_x^2 \rangle$, **[4]**

iv) and the uncertainty Δp, **[2]**

in terms of a and $\hbar$. You may use the following standard integrals:

$$\int_{-\infty}^{+\infty} e^{-bx^2}\, dx = \sqrt{\frac{\pi}{b}}, \qquad \int_{-\infty}^{+\infty} x^2 e^{-bx^2}\, dx = \frac{1}{2b}\sqrt{\frac{\pi}{b}}.$$

A6 State Heisenberg's position-momentum uncertainty relation(s) (in 3-D) in mathematical form (no words required). **[2]**

A7 Sketch the wave functions $\psi(x)$ and the probability densities $|\psi(x)|^2$ for a particle in the *first excited state of both* the infinite square well and the simple harmonic oscillator potential. **[4]**

A8 At time $t = 0$ a particle is prepared in the normalised quantum state:

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_0(x) - \frac{1}{\sqrt{2}}\,\psi_1(x),$$

where ψ_n is a normalised energy eigenstate with energy eigenvalue E_n.

i) If the system is subsequently left undisturbed what is the wave function, $\Psi(x,t)$, at a later time t? **[2]**

ii) What are the possible results of an energy measurement at time t and their respective probabilities? **[2]**

iii) Suppose a measurement yields one of the results listed in (ii); write down the wave function immediately after the measurement. **[1]**

A9 Write down an expression for the orbital angular momentum operator $\hat{L}_x$ in terms of position and momentum operators $\hat{y}, \hat{z}$ and $\hat{p}_y, \hat{p}_z$. Using this expression and the commutation relations between position and momentum operators verify the relation $[\hat{L}_x, \hat{y}] = i\hbar\hat{z}$. **[6]**

A10 A Stern-Gerlach experiment is performed with a mono-energetic beam of silver atoms. Draw a sketch of the experiment and state briefly what the apparatus is designed to measure and how this is accomplished. **[6]**

i) Why does the silver atom provide information about a single electron? **[3]**

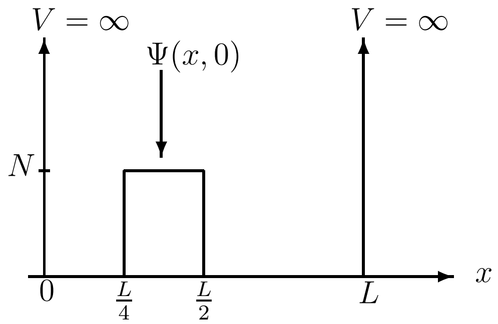
ii) A beam of spin-1 atoms is passed through the above apparatus. Show what is observed and compare this with what is seen when the magnetic field is switched off. **[2]**

SECTION B. Answer two of the four questions in this section.

B1

(a) State the expansion theorem for a particle in a 1–dimensional potential where the normalised eigenstates are $\psi_n(x)$ with corresponding eigenvalues E_n. Give a formula for the expansion coefficients c_n. **[4]**
Hence, prove that $\langle E\rangle = \langle\hat{H}\rangle = \sum_n |c_n|^2 E_n$. **[4]**

(b) At $t = 0$ a particle in an infinite square well is prepared in a state corresponding to the normalised top-hat wave function, $\Psi(x, 0)$, illustrated:



For this infinite square well potential the energy eigenstates are, for $n = 1, 2, \ldots,$

$$\begin{aligned} \psi_n(x) &= \sqrt{\tfrac{2}{L}}\,\sin(\tfrac{n\pi}{L}x) \quad \text{for} \quad 0 \le x \le L \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

i) Calculate the normalisation constant N assuming it is real and positive. **[2]**

ii) Obtain an expression for the probability that the result of an energy measurement is E_n, showing that it is proportional to $1/n^2$. Evaluate numerically the probability to find the system in the groundstate. **[6]**

(c) Consider a particle of mass m moving in a potential $V(x)$ with corresponding hermitian Hamiltonian operator $\hat{H}$ and energy eigenstates $\psi_n(x)$.

i) If the system is in any of the eigenstates ψ_n show that the expression $\langle[\hat{H}, \hat{O}]\rangle$ vanishes for any operator $\hat{O}$. **[4]**

ii) Now consider the special case $V = cx^2$ where c is a real constant and $\hat{O} = \hat{x}\hat{p}_x$. What relation between the expectation values of the kinetic and potential energies do you find? **[5]**

B2

In this question the states ψ_n are the orthonormal energy eigenstates, $\psi_n(x)$ with $n = 0, 1, 2, \ldots$, etc. of the Simple Harmonic Oscillator (SHO). You will not need explicit expressions for these eigenstates, and you may use without proof the relations $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{a}\psi_n = \sqrt{n}\psi_{n-1}$ and $\hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$.

(a) The hamiltonian of the SHO is $\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. Write down its eigenvalues E_n. **[2]**

(b) Prove the commutation relations $\left[\hat{H}, \hat{a}\right] = -\hbar\omega_0\hat{a}$ and $\left[\hat{H}, \hat{a}^\dagger\right] = +\hbar\omega_0\hat{a}^\dagger$. **[6]**

(c) Given the Heisenberg equation of motion for any operator $\hat{O}$ without explicit time-dependence

$$\frac{d\langle\hat{O}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{O}]\rangle \quad ,$$

find the Heisenberg equations of motion for $\langle\hat{a}\rangle$ and $\langle\hat{a}^\dagger\rangle$ and solve them. **[5]**

(d) Consider the SHO in the state $\psi = \psi_n$. Using the fact that $\hat{x} = \frac{1}{\sqrt{2}\beta}(\hat{a} + \hat{a}^\dagger)$ with $\beta = \sqrt{m\omega_0/\hbar}$ find $\langle x\rangle$, $\langle x^2\rangle$ and Δx using operator formalism. You may use all formulas and relations quoted in this question without proof. **[6]**

(e) Consider the SHO in its groundstate ψ_0. Calculate $\langle x^4\rangle$. **[6]**

B3

(a) Starting from the classical definition of orbital angular momentum, $\hat{\vec{L}}$, write down in a cartesian coordinate system the components $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, of the corresponding quantum mechanical operator. Hence show that they satisfy the following commutation relation: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ **[8]**

(b) The following three normalised wave functions describe possible states of an electron moving in three dimensions:

$$\begin{aligned}
\psi_1(x, y, z) &= -N(x + iy)e^{-r}, \\
\psi_2(x, y, z) &= N(x - iy)e^{-r}, \\
\psi_3(x, y, z) &= \sqrt{2}Nze^{-r}
\end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and N is a normalisation constant. Verify that each is an eigenfunction of $\hat{L}_z$ and find the corresponding eigenvalue. **[7]**
The raising angular momentum operator is defined as $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$. Show that $\hat{L}_+\psi_2 = c\hbar\psi_3$ and determine the constant c. **[4]**

(c) Describe the possible outcomes, including their probabilities, of measuring the $z-$component of the orbital angular momentum of a particle in a state with wave function:

$$\psi(x, y, z) = N(y + z)e^{-r} \ .$$

[6]

B4

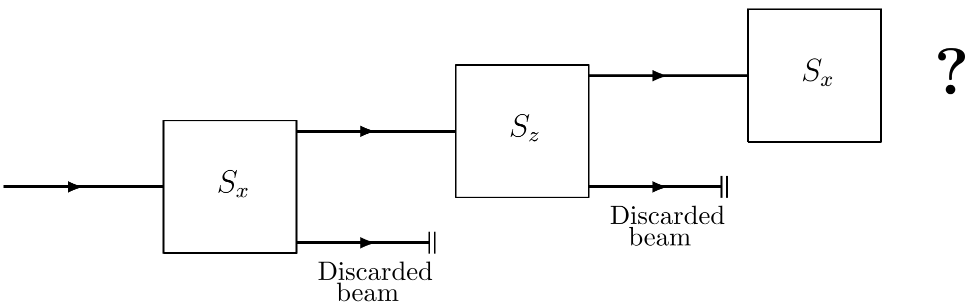
(a) Write down the matrices representing the components of the spin angular momentum operator $\widehat{\vec{S}}$ for a spin-1/2 particle. **[2]**
Write down the normalised eigenstates (eigenvectors), $\chi_{\pm}$, of $\widehat{S}_z$. **[2]**
Obtain the normalised eigenvectors, $\chi'_{\pm}$, of the operator $\widehat{S}_x$ and hence verify the following relationships between the two different sets of basis states:

$$\chi'_{\pm} = \frac{1}{\sqrt{2}}\left[\chi_+ \pm \chi_-\right] \quad \text{and} \quad \chi_{\pm} = \frac{1}{\sqrt{2}}\left[\chi'_+ \pm \chi'_-\right]$$

[6]

(b) Calculate the expectation values of $\widehat{S}_y$ and $\widehat{S}_y^2$ for spin-1/2 particles in the state χ'_+. Hence calculate the corresponding uncertainty ΔS_y. Hence, given that $\Delta S_y = \Delta S_z$, obtain the product $\Delta S_y \Delta S_z$. **[6]**
Write down the generalised Heisenberg uncertainty relation for the product $\Delta S_y \Delta S_z$. Hence, assuming the appropriate commutation relations for the spin operators, show that the state χ'_+ corresponds to the state of minimum uncertainty allowed by the Heisenberg relation. **[4]**

(c) An unpolarised beam of silver atoms enters from the left into a triple Stern-Gerlach apparatus:



Complete the diagram by labelling each beam emerging from each stage of the experiment, including the beams emerging from the final S_x Stern-Gerlach apparatus, with the appropriate normalised spin wave function *and* with the beam intensity relative to that of the incoming beam. **[5]**