Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-413 Quantum Mechanics B

Time Allowed: 2 hours 30 minutes

Date: 5^{th} May, 2010

Time: 14:30 - 17:00

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work and mid-term test comprise 20% of the final mark.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A.Brandhuber, Dr. T.Kreouzis

This (reverse cover) page is left blank.

SECTION A. Attempt answers to all questions.

A1 Write down the Time-Dependent Schrödinger Equation (TDSE) for a particle of mass m in 3-D in a potential $V(\vec{r},t)$. State under what conditions for the potential $V(\vec{r},t)$ the Time Independent Schrödinger Equation may be derived from the Time-Dependent one. Write down the Time Independent Schrödinger Equation (TISE) in 3-D. **[5]**

A2 State the Born interpretation for a normalised single particle wave function $\Psi(x,t)$ in one dimension. **[2]**

A3 What is meant by the Operator Postulate in Quantum Mechanics? **[2]**

A4 Prove the commutation relation $[\hat{x},\hat{p}_x] = i\hbar$ by using the representation $\hat{x} = x$ and $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ and verifying that the commutation relation is satisfied for any wavefunction $\Psi(x)$. Obtain an expression for the commutator $[x^2,\hat{p}_x]$. **[6]**

A5 The first excited state of the 1-D infinite square well is given by the wave function

$$\begin{aligned}\Psi_2(x,0) &= A\sin(\frac{2\pi}{L}x) \ \text{ for } -L/2 \le x \le L/2 \\ &= 0 \ \text{ elsewhere}\end{aligned}$$

Determine the constant A. (You may assume $\sin^2\theta = (1-\cos 2\theta)/2$) **[4]**

A6 Write down a general expression for the expectation value of a quantum mechanical operator $\hat{A}$ for a particle in the quantum state $\Psi(x,t)$. **[2]**

A7 Write down a general expression (ie. definition) for the uncertainty ΔA in terms of expectation values. **[2]**

A8 For a system in the state given by the wavefunction $\Psi_2(x,0)$ of Question A5 find the uncertainty Δp_x of the momentum operator $\hat{p}_x$. **[4]**

A9 State Heisenberg's position-momentum uncertainty relation(s) (in 3-D) in mathematical form (no words required). **[2]**

A10 The eigenstates $\psi_n(x)$ of a Hamiltonian are orthonormal. Write down this statement in mathematical form. **[2]**

A11 Sketch the wave functions $\psi(x)$ and the probability densities $|\psi(x)|^2$ for a particle in the *first excited state of both* the infinite square well and the simple harmonic oscillator potential. **[4]**

A12 At time $t = 0$ a particle is prepared in the normalised quantum state:

$$\Psi(x,0) = \frac{1}{2}\psi_0(x) + \frac{\sqrt{3}}{2}\,\psi_1(x),$$

where ψ_n is a normalised eigenstate with energy eigenvalue E_n.

i) If the system is subsequently left undisturbed what is the wave function, $\Psi(x,t)$, at a later time t? **[2]**

ii) What are the possible results of an energy measurement at time t and their respective probabilities? **[2]**

iii) Suppose a measurement yields one of the results listed in (ii); write down the wave function immediately after the measurement. **[1]**

A13 Write down expressions for the orbital angular momentum operator $\hat{L}_x$ in terms of position and momentum operators y, z and $\hat{p}_y, \hat{p}_z$. Using these verify the relation $[\hat{L}_x, y] = i\hbar z$. **[6]**

A14 Consider angular momentum operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$:

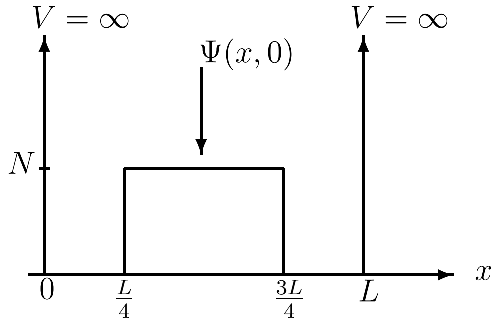
i) What are the allowed eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$? **[2]**

ii) Explain why these two operators have simultaneous eigenstates. **[2]**

SECTION B. Answer two of the four questions in this section.

B1

(a) At $t = 0$ a particle in an infinite square well is prepared in a state corresponding to the normalised top-hat wave function, $\Psi(x, 0)$, illustrated:



For this infinite square well potential the energy eigenstates are, for $n = 1, 2, \ldots$,

$$\begin{aligned} \psi_n(x) &= \sqrt{\tfrac{2}{L}}\ \sin(\tfrac{n\pi}{L}x) \quad \text{for} \quad 0 \le x \le L \\ &= 0 \quad \text{elsewhere} \end{aligned}$$

i) Calculate the normalisation constant N assuming it is real and positive. **[3]**

ii) Obtain an expression for the probability that the result of an energy measurement is E_n, showing that it falls off as $1/n^2$ with increasing n. Evaluate numerically the probability to find the system in the groundstate. **[8]**

iii) Show that for *even* n the probabilities are zero. Give a general reason for this result. **[3]**

(b) By considering the quantity,

$$\int_{-\infty}^{+\infty} \psi_i^* \widehat{H} \psi_j \, dx,$$

and assuming that the Hamiltonian is hermitian (ie. self-adjoint), prove that its eigenvalues are real and that its eigenfunctions, $\psi_n(x)$, corresponding to different eigenvalues, are orthogonal. **[7]**

(c) For a particle in a one-dimensional symmetric potential, $V(-x) = V(x)$, prove that non-degerate eigenstates have definite parity. What parity does the groundstate of any symmetric potential have? **[4]**

B2

In this question the states ψ_n, ψ_i, ψ_j are the orthonormal energy eigenstates, $\psi_n(x)$ with $n = 0, 1, 2, \ldots$, etc. of the Simple Harmonic Oscillator (SHO). You will not need explicit expressions for these eigenstates, and you may use without proof the relations $[\widehat{a}, \widehat{a}^\dagger] = 1$, $\hat{a}\psi_n = \sqrt{n}\psi_{n-1}$ and $\hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$.

(a) The hamiltonian of the SHO is $\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. Write down its eigenvalues E_n. **[2]**

(b) Using the commutation relation given above prove the commutation relations **[6]**

$$\left[\hat{H}, \hat{a}\right] = -\hbar\omega_0\hat{a} \quad \text{and} \quad \left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega_0\hat{a}^\dagger \ .$$

(c) Hence show that $\hat{a}\psi_n$ and $\hat{a}^\dagger\psi_n$ have energy eigenvalues $E_n - \hbar\omega_0$ provided that $n \geq 1$ and $E_n + \hbar\omega_0$, respectively, where E_n is the eigenvalue of ψ_n. **[5]**

(d) Remember the matrix representing an operator $\widehat{A}$ has ij-th element

$$A_{ij} = \int \psi_i^* \widehat{A}\psi_j \, dx \ .$$

Note: the matrices representing operators in the SHO problem are infinite-dimensional - write down explicitly at least the first 5×5 sub–matrix using dots to signify continuation of the pattern of entries.

(i) Show that the matrix H_{ij} representing the Hamiltonian is a diagonal matrix, and obtain its diagonal elements. **[4]**

(ii) Find the elements of the matrices representing the operators $\widehat{a}^\dagger$ and $\widehat{a}^2$. **[8]**

B3

A particle is known to be in the normalised angular momentum eigenstate

$$\psi = R(r)\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\, e^{+i\varphi} \ , \quad \text{where} \quad \int_0^\infty r^2 |R(r)|^2\, dr = 1 \ .$$

a) Show that ψ is an eigenstate of the angular momentum operators $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$ and $\hat{\vec{L}}^2 = -\frac{\hbar^2}{\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2}\right)$. Find the corresponding eigenvalues. **[7]**

b) Show explicitly by integration that the expectation value for $\widehat{L}_x$ is zero where in polar coordinates $\hat{L}_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$. **[13]**

c) What is the expectation value for a measurement of the angular momentum component $\widehat{L}_\alpha$ along a line in the $z-x$ plane rotated by an angle α away from the $z-$axis, where $\widehat{L}_\alpha = \widehat{L}_x \sin\alpha + \widehat{L}_z \cos\alpha$. **[5]**

B4

a) i) A Stern-Gerlach experiment is performed with a mono-energetic beam of silver atoms. Draw a sketch of the experiment and state briefly what the apparatus is designed to measure and how this is accomplished, showing the outcome of the experiment compared with classical expectations. **[6]**

ii) Now a beam of spin-j atoms is passed through the above aparatus. What is observed? **[2]**

b) i) Write down the matrices representing the components of the spin angular momentum operator $\hat{\vec{S}}$ for a spin-1/2 particle. Write down the eigenvectors (eigenstates), $\chi_\pm$, of $\hat{S}_z$. **[4]**

ii) Find the matrix $\hat{S}_\theta = \vec{n}\cdot\hat{\vec{S}}$ corresponding to the spin angular momentum operator projected in the direction given by the unit vector $\vec{n} = (\sin\theta, 0, \cos\theta)$. Obtain the eigenvalues of $\hat{S}_\theta$ and the normalised eigenstate χ_θ corresponding to the positive eigenvalue of $\hat{S}_\theta$. Hence, verify the relation $\chi_\theta = \cos(\theta/2)\chi_+ + \sin(\theta/2)\chi_-$. **[8]**
(Hint: you may use the identities $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$.)

iii) Calculate the expectation values of $\hat{S}_x$ and $\hat{S}_x^2$ for spin-1/2 particles in the state χ_θ. Hence calculate the corresponding uncertainty ΔS_x. **[5]**