QMB / PHY413

Course Introduction

A brief re-cap of QMA

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See Particle Data Group: "1. Physical Constants" for all physical constants quoted here. This can be found at http://pdg.lbl.gov/ K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

Introduction

Plank: Quantization

- 1900: Plank introduced the concept that electromagnetic radiation is emitted and absorbed as quanta of energy.

$$\begin{aligned} E &= h\nu \\ &= \hbar\omega \end{aligned}$$

Recall:

$$\hbar = h/2\pi$$

$$\begin{aligned} \hbar &= 1.054571628(53) \times 10^{-34} Js \\ &= 6.58211899(16) \times 10^{-22} MeVs \\ &= 1\,\text{natural units} \end{aligned}$$

  - Based on theoretical work by Stefan and Boltzmann on Blackbody radiation, and detailed experimental work (see note set 1).

Refs. [1] B&J Chapter 1
[2] Feynman Lectures, Volume III. Chapter 2

Einstein: Photoelectric effect

- 1905: The relativistic behaviour led to mass-energy equivalence via

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

- The photoelectric effect could be explained using this, where (as photons are massless)

$$E = pc$$

Refs. [1] B&J Chapter 1

Wave-particle duality

- 1923: A step forward in comprehension
  - Particles can behave as waves
  - Waves can behave as particles
    - There is an intrinsic scale to the problem defined by the energy one is working at.

$$\begin{aligned} p &= \frac{h}{\lambda} \\ &= \hbar k \end{aligned}$$

k = wave number

  - So for all particles one finds

$$E = \sqrt{p^2c^2 + m_0^2c^4} \equiv \hbar\omega$$

  - e.g. a plane wave can be expressed as

$$\Psi(x,t) = e^{i(kx-\omega t)} = e^{i(px-Et)/\hbar}$$

Refs. [1] B&J Chapter 1
[2] Feynman Lectures, Volume III. Chapter 2

The Wave Eqn. TDSE/TISE

- From energy considerations:

$$\begin{aligned} E &= KE + PE \\ &= \frac{p^2}{2m} + V(x,t) \end{aligned}$$

- From this one deduces:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

- If the potential is time-independent, then we separate out the wave function into space and time parts:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = -\frac{iE}{\hbar}f(t)$$

- where

$$\Psi(x,t) = \psi(x)f(t)$$

Refs. [1] B&J Chapter 3

Some observations

- The Hamiltonian operator is Ĥ:

$$\widehat{H}\psi(x) = E\psi(x)$$

- The solution $\psi_E(x)$ is the eigenfunction of Ĥ with energy eigenvalue E.

- E is real and for the TISE Ĥ is real, so the $\psi_E(x)$ are real.
  - Eigenvalues of physical observables are real.
  - This does not hold for the TDSE, as the energy is real, but time-dependence is complex.

Superposition principle

- In general the wavefunction is given by

$$\Psi(x,t) = \sum_{i=1}^{\infty} c_i \psi_i(x,t)$$

- This is a sum over all possible independent solutions of the wave equation.

- The (in general complex) coefficients c_i must satisfy the overall normalisation condition dictated by the Born interpretation (next slide)

Born interpretation

- The wave function is a probability amplitude. i.e.

$$\mathcal{P} = |\Psi(x,t)|^2 \mathrm{d}x$$

  - The probability that a measurement at some time t of the particle position x will yield a result between x and x+dx.
  - In 3D this becomes

$$\begin{aligned} \mathcal{P} &= |\Psi(\underline{r},t)|^2 \mathrm{d}^3x \\ \mathrm{d}^3x &= \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned}$$

- The probability of finding the particle anywhere given a measurement at time t should be 1.
  - So we have our normalisation condition:

$$\int_{-\infty}^{\infty} |\Psi(\underline{r},t)|^2 \mathrm{d}^3x = 1$$

Question

- If I have a wave-function given by

$$\Psi = Ae^{i\phi}$$

  - What is this (basic terms – what type of quantity)?
  - What does it represent?
  - What is the probability of finding the particle at a given point ϕ?
  - How does the probability vary with ϕ?

TISE

- Revisit the TISE: V(x,t) = V(x), so

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+V(x)\psi(x)=E\psi(x) \qquad \frac{\mathrm{d}f(t)}{\mathrm{d}t}=-\frac{iE}{\hbar}f(t)$$

and

$$\Psi(x,t)=\psi(x)f(t)$$

ψ(x) depends on the potential, whereas only the RHS is time-dependent. We can solve for f(t)

$$f(t)=Ae^{-iEt/\hbar}$$

so

$$\Psi(x,t)=\psi(x)e^{-iEt/\hbar}$$

The normalisation constant has been absorbed into ψ(x), and we can now use this solution in the SE.

Refs. [1] B&J Chapter 3

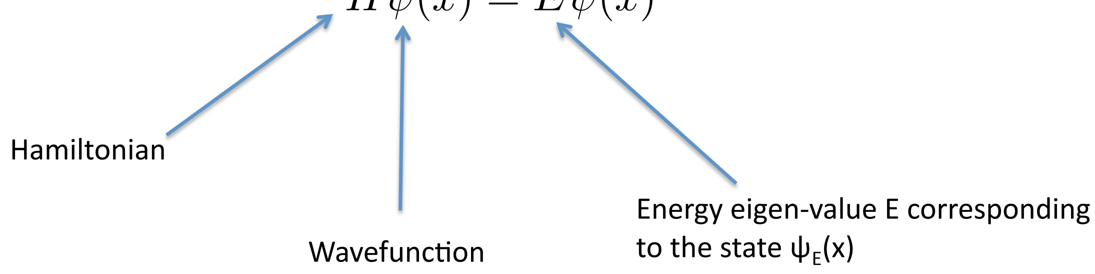
TISE

- so

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+V(x)\psi(x)=E\psi(x)$$

can be written as

$$\widehat{H}\psi(x)=E\psi(x)$$



Hamiltonian

Wavefunction

Energy eigen-value E corresponding to the state $\psi_E(x)$

E is real, so ψ_E is real

but $\Psi(x,t) = \psi_E\, e^{-iEt/\hbar}$ is complex

Refs. [1] B&J Chapter 3

Consequences of the Born interpretation

- The wavefunction encodes everything we are able to determine about a state.
- The measurement of a quantity (e.g. position) is probabilistic, and not a deterministic one.
- We can prepare a state Ψ(x,t), and subsequently make a measurement of the position at some time t.
  - The wave function tells is the probability of observing the state between x and x+dx is given by

$$\mathcal{P} = |\Psi(x,t)|^2 \mathrm{d}x$$

Consequences of the Born interpretation

- We can prepare an ensemble of states Ψ(x,t), and subsequently measure them.
- From the ensemble we can compute an expectation value in the normal way

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 \mathrm{d}x$$

- We normally write this as

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \Psi^* x \Psi \mathrm{d}x \end{aligned}$$

Consequences of the Born interpretation

- This is the ensemble (or Copenhagen) interpretation of QM.
- We can compute a spread on x given by Δx, where

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

- We know <x> (previous slide), and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \mathrm{d}x$$

- so Δx is calculable.

Two slit experiment revisited

- Prepare an ensemble of states

$$\Psi = \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)$$

where

$$\begin{aligned} |\Psi|^2 &= \frac{1}{2}(|\Psi_1|^2 + |\Psi_2|^2 + \Psi_1^*\Psi_2 + \Psi_1\Psi_2^*) \\ &= \frac{1}{2}(|\Psi_1|^2 + |\Psi_2|^2 + 2Re[\Psi_1^*\Psi_2]) \end{aligned}$$

Which slit did a particle go through?

Can't tell as we didn't measure that.

The wavefunction Ψ represents an ENSEMBLE of states with equal probability amplitude that a given particle is described by either Ψ_1 or by Ψ_2.

Interference ($\Psi_i{}^*\Psi_j$, i≠j) is the key feature here.

Boundary conditions

1. Physical states are localised, so

$$\lim_{x \to \pm\infty} \psi(x) = 0$$

2. The WF Ψ must be square integrable (have to be able to normalise it).

3. $|\Psi|^2$ is a probability density function, so it has to be single valued.
   1. But the wave function Ψ does not have to satisfy this.

4. The wave function has to be continuous everywhere (follows point 3).

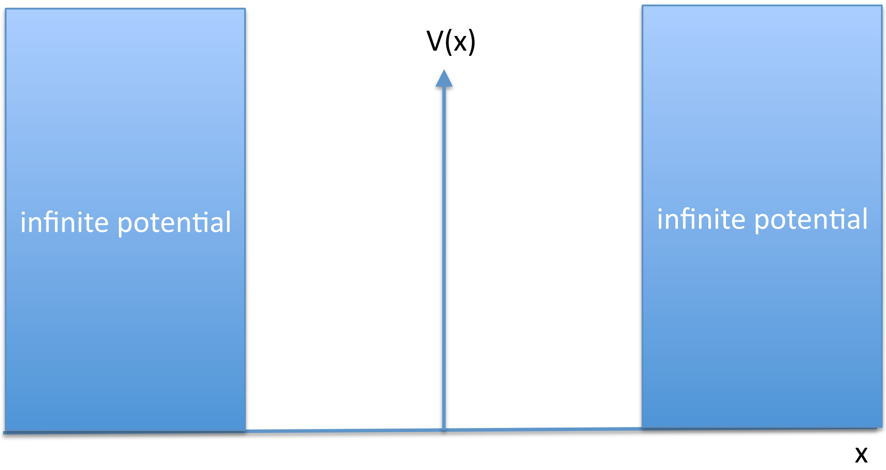
5. The derivative dΨ/dx has to be continuous everywhere so that the second derivative is finite in the Hamiltonian.
   1. Exception is the infinite square well example.

Infinite Square Well

- V(x) is independent of time, so

$$\Psi(x,t) = \psi(x) e^{-iEt/\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$



Infinite Square Well

- V(x) is independent of time, so

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \tag{1}$$

- Note that:

$$\begin{aligned} \psi(x) &= 0 \text{ when } x = \pm L/2 \\ \psi(x) &= 0 \text{ when } |x| > L/2 \\ V(x) &= 0 \text{ when } |x| < L/2 \end{aligned}$$

- So Eq. (1) gives

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -k\psi(x), \text{ where } k = \sqrt{\frac{2mE}{\hbar}}$$

- Solutions take the form

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

- Odd and even parts to the probability amplitude.

Infinite Square Well

- Solutions are given by

$$E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}, \text{ where } n = 1,2,3,4,...$$

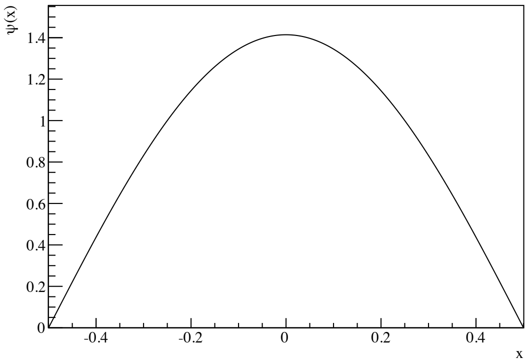
$$\psi(x) = \sqrt{\frac{2}{L}}\left[\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\right]$$

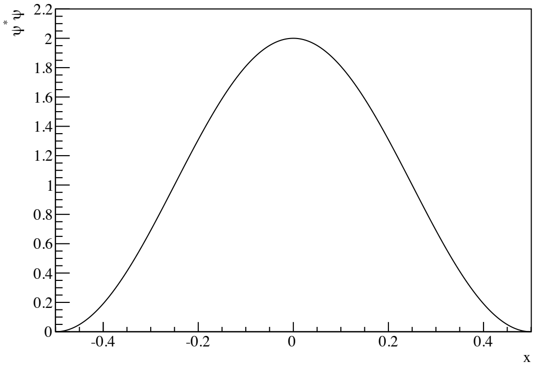
non-vanishing for odd values of n
(i.e. ignore term for even n)

non-vanishing for even values of n
(i.e. ignore term for odd n)

Infinite Square Well

- n=1: Ground state

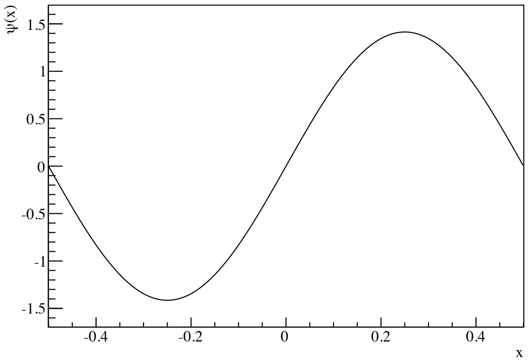

Wave function (un-normalised)

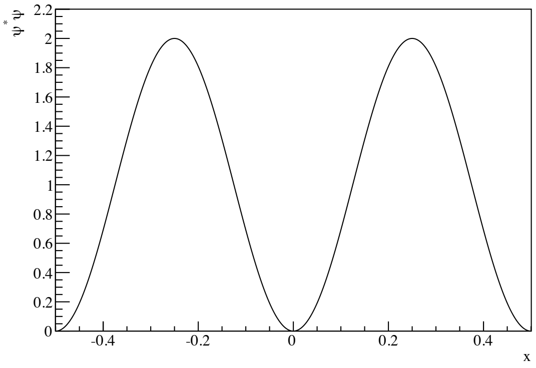

PDF (un-normalised)

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)e^{-iE_1 t/\hbar},\ \text{and}\ E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

Infinite Square Well

- n=2: First excited state


Wave function (un-normalised)

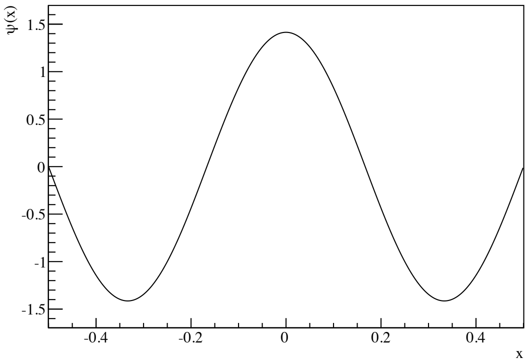

PDF (un-normalised)

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)e^{-iE_2 t/\hbar},\ \text{and}\ E_2 = \frac{4\pi^2\hbar^2}{2mL^2}$$

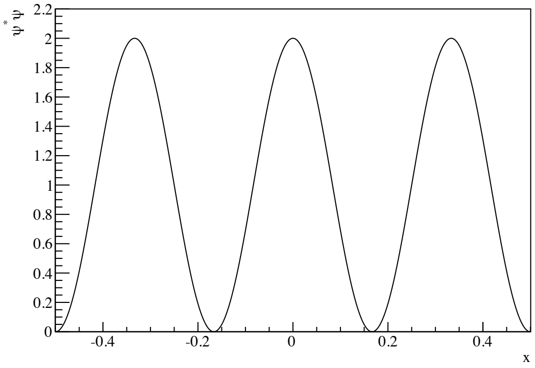
Infinite Square Well

- n=3: Second excited state

Wave function (un-normalised)



PDF (un-normalised)

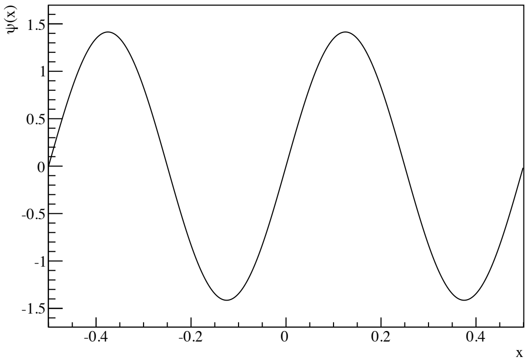


$$\Psi(x,t) = \sqrt{\frac{2}{L}} \cos\left(\frac{3\pi x}{L}\right) e^{-iE_3 t/\hbar}, \text{ and } E_3 = \frac{9\pi^2\hbar^2}{2mL^2}$$

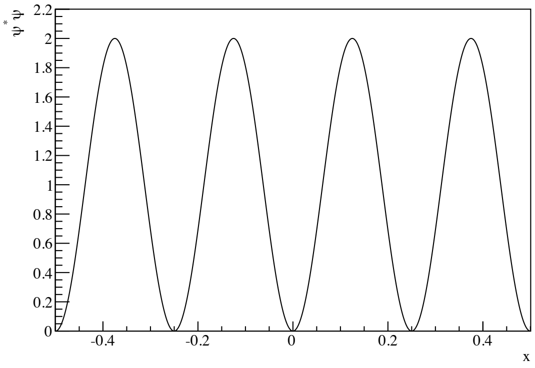
Infinite Square Well

- n=4: Third Excited state

Wave function (un-normalised)



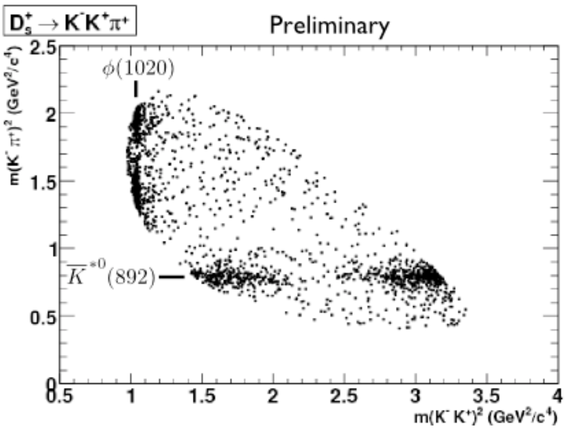
PDF (un-normalised)



$$\Psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{4\pi x}{L}\right) e^{-iE_4 t/\hbar}, \text{ and } E_4 = \frac{16\pi^2\hbar^2}{2mL^2}$$

Question

- This is a Dalitz plot. What do you think is giving this structure?



Just like the finite square well potential, but here the quantum degree of freedom is something called spin. We'll come back to that later.

See http://superweak.wordpress.com/2006/07/31/dalitz-plots/, and for a more detailed description refer to the PDG section on Kinematics.

Aside

- The K*(892) and ϕ(1020) particles are $J^P = 1^-$.
  - These are vector particles. Here we have

$$\phi(1020) \to K^+K^-$$
$$K^{*0}(892) \to K^+\pi^-$$

- The spin-one (J=1) quantum number assignment is analogous to the first excited state (n=2) of the ISW problem.

- We will see later that P=−1 indicates that the wave-function is also anti-symmetric (like the n=2 ISW solution).

- See PDG Kinematics review for more information on 3 body decays.

Momentum

- Consider a de Broglie wave as an idealised example:

$$\Psi(x,t) = e^{i(kx-\omega t)} = e^{i(px-Et)/\hbar}$$

- Normalisation of the WF is a problem, however we can compute a wave packet consisting of an infinite sum of possible de Broglie waves to obtain a momentum wave function Φ(p,t), where

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{i(px-Et)/\hbar} \mathrm{d}p$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p,t) e^{ipx/\hbar} \mathrm{d}p$$

- and an inverse Fourier Transform yields

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ipx/\hbar} \mathrm{d}x$$

Momentum

- Interpret |Φ|^{2}dp as the probability of observing a particle with momentum between p and dp.
- It follows (see Appendix of Note Set 1) that

$$\langle p \rangle = \int_{-\infty}^{\infty} \Phi^*(p,t)\, \widehat{p}\, \Phi(p,t) \mathrm{d}p$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x,t) \mathrm{d}x$$

- From this interpretation we can see that

$$\widehat{p} \equiv -i\hbar \frac{\partial}{\partial x}$$

- where we explicitly denote this as an operator (like the Hamiltonian earlier)

Operators

- Dynamical variables in QM are obtained by replacing their classical momentum variables with the QM operators.
  - e.g. $A(x,p) \to \widehat{A} = A(x,\widehat{p})$
  - so

$$\begin{array}{rcl} x & \to & x \\ p & \to & \widehat{p} = -i\hbar\dfrac{\partial}{\partial x} \\ H & \to & \widehat{H} \\ L = \underline{r}\times\underline{p} & \to & \widehat{L} = \underline{r}\times\widehat{\underline{p}} \\ & & \text{so } \widehat{L} = i\hbar\underline{r}\times\nabla \end{array}$$

  - Hence the SE equation becomes:

$$\widehat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

Heisenberg's Uncertainty Principle

- Stems from commutation relations (c.f. the non-commutability of matrices).
- The commutator of two operators is given by

$$[A,B] = AB - BA$$

- e.g. differentiation operators do not commute

$$[x,\widehat{p}_x] = i\hbar$$

- as

$$\begin{array}{rcl} [x,\widehat{p}_x]\psi & = & (x\widehat{p}_x - \widehat{p}_x x)\psi \\ & = & -i\hbar\left[x\dfrac{\partial}{\partial x}\psi(x) - \dfrac{\partial}{\partial x}\{x\psi(x)\}\right] \\ & = & -i\hbar\left[x\dfrac{\partial}{\partial x}\psi(x) - x\dfrac{\partial}{\partial x}\psi(x) - \psi(x)\right] \\ & = & i\hbar\psi(x) \end{array}$$

Heisenberg's Uncertainty Principle

- A general argument can be used to show that for any two operators

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\widehat{A}, \widehat{B}] \rangle \right|$$

- where ΔA and ΔB are the respective uncertainties on an ensemble measurement of the two observables on a system in a given quantum state.

- For momentum and space we find

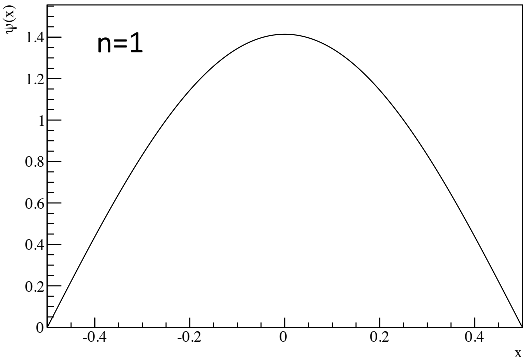
$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

- i.e. it is not possible to simultaneously know x and p_x to infinite precision.

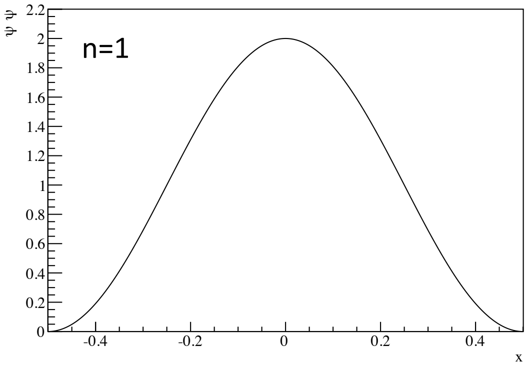
Energy eigenstates

- Consider solutions to the TISE i.e. V(x,t) = V(x)
  - e.g. the infinite square well (reminder of the ground state)

Wave function (un-normalised)



PDF (un-normalised)



$$\Psi(x,t) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) e^{-iE_1 t/\hbar}, \text{ and } E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$$

Energy eigenstates

- Solutions have the form:

$$\Psi_E(x,t) = \psi_E(x)e^{-iEt/\hbar}$$

- hence

$$\Psi^*_E(x,t) = \psi^*_E(x)e^{+iEt/\hbar}$$

- So the probability is given by

$$\begin{aligned} P(x,t) &= \Psi^*_E(x,t)\Psi_E(x,t) \\ &= |\Psi_E(x,t)|^2 \\ &= |\psi_E(x)|^2 \\ &= P(x) \end{aligned}$$

- i.e. probability is independent of time...
- So the state Ψ_E should be stationary.

Energy eigenstates

- For solutions of the TISE, the time dependence falls out of the expectation value computation.
  - See notes for general discussion.
- 1) let's consider momentum: $\widehat{p} = -i\hbar\dfrac{\partial}{\partial x}$

$$\langle\widehat{p}\rangle = -i\hbar\int\limits_{-\infty}^{+\infty}\psi^*_n(x)\frac{\partial}{\partial x}\psi_n(x)\mathrm{d}x$$

- So the complex conjugate of this is

$$\begin{aligned} (\langle\widehat{p}\rangle)^* &= \left[-i\hbar\int\limits_{-\infty}^{+\infty}\psi^*_n(x)\frac{\partial}{\partial x}\psi_n(x)\mathrm{d}x\right]^* \\ &= i\hbar\int\limits_{-\infty}^{+\infty}\psi_n(x)\frac{\partial}{\partial x}\psi^*_n(x)\mathrm{d}x \\ &= -\langle\widehat{p}\rangle \end{aligned}$$

<p>=0 is the only valid solution as the expectation value of an observable has to be a real quantity.

Energy eigenstates

- 2) now consider energy expectation values and the Hamiltonian

$$\begin{aligned}\langle\widehat{H}\rangle &= \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\widehat{H}\Psi_n(x,t)\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \psi_n^*(x)\widehat{H}\psi_n(x)\mathrm{d}x \\ &= E_n\end{aligned}$$

- Energy is quantised so we would anticipate that the uncertainty on E would be zero, where

$$\Delta E = \sqrt{\langle\widehat{H}^2\rangle - \langle\widehat{H}\rangle^2}$$

- This is left as an exercise for you to show.

Orthogonality and Orthonormality

- It can be shown (see note set 1) that

$$\int_{-\infty}^{+\infty} \Psi_i^*\widehat{H}\Psi_j\mathrm{d}x = \int_{-\infty}^{+\infty} (\widehat{H}\Psi_i)^*\Psi_j\mathrm{d}x$$

- which can be used to show that the energy eigenstates are orthogonal.

$$\begin{aligned}H_{ij} &= \int_{-\infty}^{+\infty} \Psi_i^*\widehat{H}\Psi_j\mathrm{d}x = E_j\int_{-\infty}^{+\infty} \Psi_i^*\Psi_j\mathrm{d}x & \text{(1)} \\ &= \int_{-\infty}^{+\infty} (\widehat{H}\Psi_i)^*\Psi_j\mathrm{d}x = E_i^*\int_{-\infty}^{+\infty} \Psi_i^*\Psi_j\mathrm{d}x & \text{(2)}\end{aligned}$$

- So (1) – (2) gives

$$(E_j - E_i^*)\int_{-\infty}^{+\infty} \Psi_i^*\Psi_j\mathrm{d}x = 0$$

Orthogonality and Orthonormality

- Case 1: i=j

$$(E_j - E_i^*) \int_{-\infty}^{+\infty} \Psi_i^* \Psi_j \mathrm{d}x = 0$$

  - E_i-E_i*=0 given by the requirement that the energy eigen values must be real.
  - The integral is simply the familiar normalisation condition

$$\int_{-\infty}^{\infty} \Psi_i^* \Psi_i \mathrm{d}x = 1$$

Orthogonality and Orthonormality

- Case 2: i≠j

$$(E_j - E_i^*) \int_{-\infty}^{+\infty} \Psi_i^* \Psi_j \mathrm{d}x = 0$$

  - E_j-E_i*≠0 given non-degenerate solutions.
  - Hence the integral must be zero!

$$\int_{-\infty}^{\infty} \Psi_i^* \Psi_j \mathrm{d}x = 0, \mathrm{for}\ i \neq j$$

  - Hence the energy eigenstates are orthogonal.

Orthogonality and Orthonormality

- We can encode these two scenarios into a single expression: the orthonormality condition

$$\int_{-\infty}^{\infty} \Psi_i^* \Psi_j \mathrm{d}x = \delta_{ij}$$

- where the RHS is the Kronecker delta:

$$\begin{aligned} \delta_{ij} &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j \end{aligned}$$

An admixture of states: example

- Consider a cold gas of hydrogen atoms in their ground state.
- If the gas is excited by a laser such that there is a 50/50 probability that an atom moves from the ground state (n=1) to the first excited state (n=2), then

$$\begin{aligned} \Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\Psi_1(x,t) + \Psi_2(x,t)\right] \\ &= \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right] \end{aligned}$$

- The time-dependence does not vanish from the probability density function:

$$\begin{aligned} \Psi^*(x,t)\Psi(x,t) &= \frac{1}{2}\left[|\Psi_1(x,t)|^2 + |\Psi_2(x,t)|^2 + \Psi_1^*(x,t)\Psi_2(x,t) + \Psi_2^*(x,t)\Psi_1(x,t)\right] \\ &= \frac{1}{2}\left[|\Psi_1(x,t)|^2 + |\Psi_2(x,t)|^2 + 2Re\left\{\Psi_1^*(x,t)\Psi_2(x,t)\right\}\right] \\ &= \frac{1}{2}\left[|\Psi_1(x,t)|^2 + |\Psi_2(x,t)|^2 + 2\psi_1(x)\psi_2(x)\cos\left(\frac{E_2 - E_1}{\hbar}t\right)\right] \end{aligned}$$

An admixture of states: example

- The angular frequency depends on the energy difference

$$\omega_{12} = \frac{E_2 - E_1}{\hbar}$$

- But the normalisation condition is necessarily constant:

$$\int\limits_{-\infty}^{\infty} |\Psi|^2 \mathrm{d}x = 1$$

An admixture of states: example

- Consider the energy expectation value:

$$\begin{aligned} \langle \widehat{H} \rangle &= \int\limits_{-\infty}^{\infty} \Psi^* \widehat{H} \Psi \mathrm{d}x \\ &= \frac{1}{2} \int\limits_{-\infty}^{\infty} \Psi_1^* \widehat{H} \Psi_1 + \Psi_2^* \widehat{H} \Psi_2 \mathrm{d}x \\ &= \frac{E_1 + E_2}{2} \end{aligned}$$

- And the uncertainty on E is given by

$$\Delta E = \sqrt{\langle \widehat{H}^2 \rangle - \langle \widehat{H} \rangle^2}$$

- It can be shown (left as an exercise) that:

$$\Delta E = \frac{E_2 - E_1}{2}$$

An admixture of states: example

- In general one can consider the case where the WF is

$$\Psi = \sum_i c_i \Psi_i$$

- e.g. for 2 components

$$\Psi = c_1 \Psi_1 + c_2 \Psi_2$$

- where normalisation requires that

$$|c_1|^2 + |c_2|^2 = 1$$

- and the expectation values are modified accordingly.
  - This formalism is applicable to a number of physical situations
    - Stern Gerlach experiment
    - Neutral K, D, B_d and B_s mixing
    - neutrino oscillation
    - Quantum information algorithms
    - etc.

The measurement postulate

- The most general solution for the TDSE is a linear combination of all possible energy eigenfunctions:

$$\begin{aligned}\Psi &= \sum_n c_n \Psi_n \\ &= \sum_n c_n \psi_n e^{-iE_n t/\hbar}\end{aligned}$$

- where the coefficients are given by

$$c_n = \int_{-\infty}^{+\infty} \Psi(x,0)\psi_m^*(x)\mathrm{d}x$$

(See notes for proof)

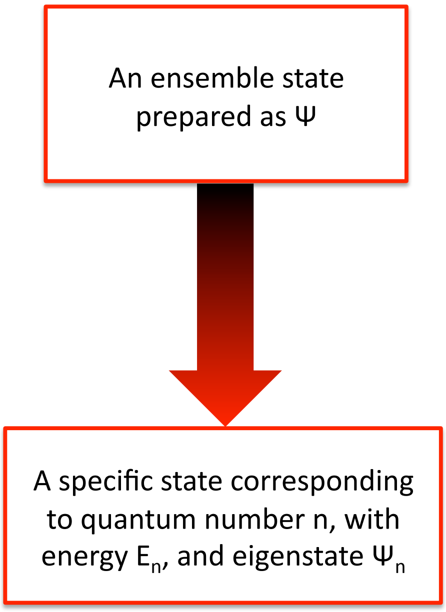
- What do the c_n represent?
  - Consider the energy expectation value <E> to develop an insight on this question.

The measurement postulate

$$
\begin{aligned}
\langle\widehat{H}\rangle &= \int_{-\infty}^{\infty} \Psi^* \widehat{H} \Psi \mathrm{d}x \\
&= \sum_m \sum_n c_m^* c_n e^{i(E_m - E_n)t/\hbar} \int_{-\infty}^{\infty} \psi_m^* \widehat{H} \psi_n \mathrm{d}x \\
&= \sum_m \sum_n c_m^* c_n E_n e^{i(E_m - E_n)t/\hbar} \int_{-\infty}^{\infty} \psi_m^* \psi_n \mathrm{d}x \\
&\left.\begin{aligned}
&= \sum_m \sum_n c_m^* c_n E_n e^{i(E_m - E_n)t/\hbar} \delta_{mn} \\
&= \sum_n |c_n|^2 E_n
\end{aligned}\right\} \text{Using orthonormality}
\end{aligned}
$$

- We interpret $|c_n|^2$ as the probability of observing the system in state n, with energy E_n subsequent to a measurement.

The measurement postulate



An ensemble state prepared as Ψ

A specific state corresponding to quantum number n, with energy E_n, and eigenstate Ψ_n

$$\Psi = \sum_n c_n \Psi_n$$

The wave function collapses from the ensemble of possible states into a definite state.

The collapse is triggered by the act of making a measurement.

$$\Psi_n$$

Dirac notation

- We will see a simplified notation during this course

$$\int_{\text{all space}} \Psi^* \Psi \mathrm{d}x = \langle \Psi | \Psi \rangle$$

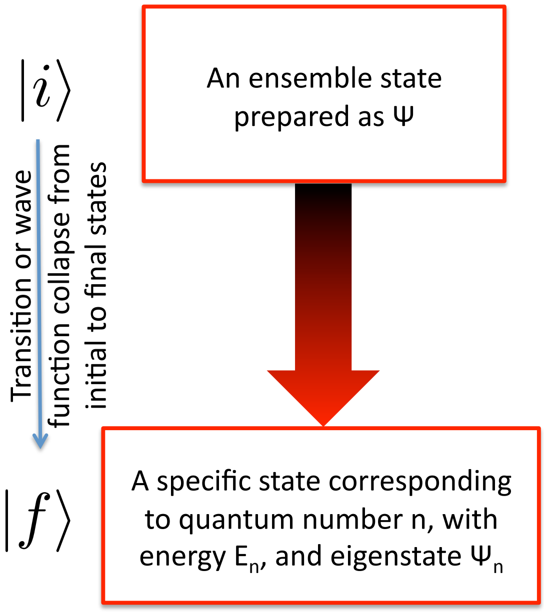
$$\int_{\text{all space}} \Psi^* \widehat{O} \Psi \mathrm{d}x = \langle \Psi | \widehat{O} | \Psi \rangle$$

- This is known as the Dirac notation (or Dirac braket notation). Here

$$\begin{aligned} \text{bra} &= \langle \Psi | \\ \text{ket} &= | \Psi \rangle \end{aligned}$$

which is a useful shorthand for the integral equations.

The measurement postulate



$|i\rangle$

An ensemble state prepared as Ψ

Transition or wave function collapse from initial to final states

$|f\rangle$

A specific state corresponding to quantum number n, with energy E_n, and eigenstate Ψ_n

$$\Psi = \sum_n c_n \Psi_n$$

The wave function collapses from the ensemble of possible states into a definite state.

The collapse is triggered by the act of making a measurement.

$$\Psi_n$$