Answer 1 (i) [6 marks] Recall Gauss' law $\int_V \nabla \mathbf{X}\, dV = \int_S \mathbf{X} \cdot d\mathbf{S}$, for any vector field $\mathbf{X}$. Let C be the closed contour in space with line element $d\mathbf{l}$ along the contour. We also have Stokes' theorem $\oint_C \mathbf{X} \cdot d\mathbf{l} = \int_{S'} (\nabla \times \mathbf{X}) \cdot d\mathbf{S}'$. These results may be used to write Maxwell's equations as given. These follow by integrating the scalar Maxwell equations over a volume V and the vector equations over a surface S' with element $d\mathbf{S}'$.

(ii) [8 marks] Consider first a very small shallow cylinder which straddles the boundary between the two regions, for which the normals to the circular ends of the cylinder are perpendicular to the boundary. Apply the first and third of the Maxwell relations above to the volume and surface of this cylinder. Ignoring the contribution from the infinitely thin sides of the cylinder one finds that $\oint_S \mathbf{D} \cdot d\mathbf{S} = (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} \Delta a$, where Δa is the area of the circular end of the cylinder, and $\mathbf{n}$ the unit normal to the boundary. For the electric case, given a surface charge density σ we have $\int_V \rho\, dV = \sigma \Delta a$. Thus we deduce the boundary conditions

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma, \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0,$$

the second equation following by similar arguments, noting the absence of magnetic charges.

Now consider a small rectangle which straddles the boundary between the two media. This rectangle has short sides which are infinitesimally small, and longer sides of length Δl which are parallel to the boundary. The unit normal $\mathbf{t}$ to the rectangle is tangent to the interface between the regions. Then $\oint_C \mathbf{H} \cdot d\mathbf{l} = (\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{H}_2 - \mathbf{H}_1) \Delta l$. Assume that there is a current density $\mathbf{K}$ flowing on the rectangle surface $\mathbf{S}'$. Then $\int_{S'} (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S}' = \mathbf{K} \cdot \mathbf{t} \Delta l$, since the $\mathbf{D}$ term vanishes as the area of the cylinder goes to zero. Thus we deduce from the second and fourth of the Maxwell integral relations that

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}, \quad \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0,$$

the second equation following again by the same arguments, noting the absence of magnetic sources.

(iii) [6 marks]

The relevant boundary condition is

$$\mathbf{E}_p + \mathbf{E}''_p = \mathbf{E}'_p,$$

where the subscript p refers to the component parallel to the interface. This immediately gives the required equation. If this is true for all x then the exponents in the equation must be equal, which implies the two relations given.

Answer 2

(i) [4 marks] Use $\mathbf{B} = \nabla \times \mathbf{A}$, and $\nabla r = \mathbf{n}$ and note that the ∇ operator when acting on the $1/r$ term produces a term of higher order in $1/kr$ in the far zone. Then the curl just produces an expression $ik\ \mathbf{n}\times$ which gives the result.

(ii) [5 marks] From the fact that the time dependence of the fields is $e^{-i\omega t}$ one has $\dot{\mathbf{E}} = -i\omega\mathbf{E}$. Then, using the same arguments as in part (i) one deduces from the given source-free equation that in the far zone one obtains the required expression.

(iii) [8 marks] The first part is immediate, using $c^2 = 1/\epsilon_0\mu_0$.

Then

$$\mathbf{B}^{m.d.}(\mathbf{m}) = \nabla\times\mathbf{A}^{m.d.}(\mathbf{m}) = \frac{i}{kc}\nabla\times\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}) = \frac{1}{c^2}\mathbf{E}^{e.d.}(\mathbf{p}\to\mathbf{m}).$$

Also,

$$\mathbf{E}^{m.d.}(\mathbf{m}) = \frac{ic}{k}\nabla\times\mathbf{B}^{m.d.}(\mathbf{m}) = \frac{i}{kc}\nabla\times\mathbf{E}^{e.d.}(\mathbf{p}\to\mathbf{m}) = -\frac{1}{k^2}\nabla\times(\nabla\times\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m})).$$

This equals

$$\frac{1}{k^2}\nabla^2\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}) = \frac{1}{k^2}\nabla\times(\nabla^2\mathbf{A}^{e.d.}(\mathbf{p}\to\mathbf{m})) = -\frac{\omega^2}{k^2c^2}\nabla\times\mathbf{A}^{e.d.}(\mathbf{p}\to\mathbf{m})$$

$$= -\mathbf{B}^{(e.d.)}(\mathbf{p}\to\mathbf{m}).$$

(iv) [3 marks] The polarisation is defined as the direction of the electric field $\mathbf{E}$. For the electric dipole $\mathbf{E}, \mathbf{p}$ and $\mathbf{n}$ are in the same plane, and $\mathbf{B}$ is perpendicular to this plane, with $\mathbf{E}, \mathbf{B}, \mathbf{n}$ mutually perpendicular.

For the magnetic dipole $\mathbf{E}, \mathbf{m}$ and $\mathbf{n}$ are in the same plane, and $\mathbf{B}$ is perpendicular to this plane, with $\mathbf{E}, \mathbf{B}, \mathbf{n}$ mutually perpendicular.

Thus the two are related by interchanging $\mathbf{E}, \mathbf{B}$ and interchanging $\mathbf{p}, \mathbf{m}$.

Answer 3

(i) [3 marks] We have $\partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu$ in Lorentz gauge, from which it is straightforward to deduce the two equations.

(ii) [4 marks] This requires integrating the term involving $\frac{\partial^2}{\partial t^2}$ twice by parts to bring down a factor of $-\omega^2$ from the exponential, and dropping boundary terms assuming that the fields and their first two time derivatives fall off to zero at infinity.

(iii) [3 marks] Here one pulls the d'Alembertian operator inside the integral and acts with it on G. This generates a delta function which is then integrated with $\mathbf{J}$ to give the required answer.

(iv) [5 marks] The d'Alembertian operator is invariant under translations and spatial rotations, hence the function G must be a function of the scalar r alone. When $r \neq 0$, the delta function does not contribute and one has a standard second order ordinary differential equation for rG, which is solved by an arbitrary linear combination of the two exponentials.

(v) [5 marks] When $r \to 0$, then the $1/r$ term dominates on the left-hand side and one has

$$\frac{1}{r}\frac{d^2}{dr^2}(rG) = -4\pi\delta^3(\mathbf{r}).$$

This is Poisson's equation if one identifies $\Phi = G_k$, and $\rho = 4\pi\epsilon\delta^3(\mathbf{r})$. Using this in the given solution one finds that $G = 1/r$ and hence that one must have $A + B = 1$.

Answer 4 (i) [6 marks] These follow from *div curl* $= 0 =$ *curl grad* and manipulations of the equations.

(ii) [2 marks] Straightforward calculation.

(iii) [4 marks] This follows as the partial derivatives commute. The gauge transformations on A_μ are

$$A_\mu \to A_\mu - \partial_\mu \Lambda$$

which leave $F_{\mu\nu}$ invariant as the derivatives commute again.

(iv) [8 marks] Write

$$\mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' = \nabla^2 \int \frac{-1}{4\pi} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}$$

$$= -\frac{1}{4\pi}\nabla\nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' + \frac{1}{4\pi}\nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' =: \mathbf{J}_l + \mathbf{J}_t$$

with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$, so that these fields are longitudinal and transverse respectively. Now $\nabla^2 \Phi = -\frac{1}{\epsilon_0}\rho$ in the Coulomb gauge, so that

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}'$$

whence

$$\frac{1}{c^2}\,\nabla\dot{\Phi} = \frac{1}{4\pi\epsilon_0 c^2}\nabla \int \frac{\dot{\rho}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = -\frac{\mu_0}{4\pi}\nabla \int \nabla' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = \mu_0 \mathbf{J}_l$$

(as $\nabla \cdot \mathbf{J} + \dot{\rho} = 0$). Thus

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla\dot{\Phi} = -\mu_0 \mathbf{J}_t.$$

Answer 5 (i) [2 marks] The first term in the expression for the electric field (the velocity field) may be dropped, since we are interested only in the field far from the particle, and the acceleration field dominates there. Furthermore, we are told that the acceleration is parallel to the velocity, so that $\beta \times \dot{\beta} = 0$, and what is left is what was asked to be shown.

(ii) [4 marks] The magnetic field may be derived from $\mathbf{B} = [\mathbf{n} \times \mathbf{E}]_{\rm ret}/c$, and then we have

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\Big(\frac{q}{4\pi\epsilon_0 c}\Big)^2 \Big[\frac{\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\Big]_{\rm ret} \times \Big\{\mathbf{n}\times\Big[\frac{\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\Big]_{\rm ret}\Big\},$$

and after use of the identity

$$\Big(\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]\Big)\times\Big(\mathbf{n}\times\big(\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]\big)\Big) = [\mathbf{n}\times\dot{\beta}]^2\mathbf{n}$$

this gives the desired result.

(iii) [6 marks] With the expression just obtained, $\mathbf{n}\cdot\mathbf{S}$ is the energy per unit area per unit time detected at the observation point at the time t. This was emitted by the particle at the retarded time $t' = t - R(t')/c$. So in a time interval from $t' = T_1$ to time $t' = T_2$ the energy radiated would be

$$E = \int_{t=T_1+R(T_1)/c}^{t=T_2+R(T_2)/c} \mathbf{n}\cdot\mathbf{S}\,dt = \int_{t'=T_1}^{t'=T_2} \mathbf{n}\cdot\mathbf{S}\frac{dt}{dt'}\,dt',$$

which implies the required result. Using $\frac{dt}{dt'} = 1-\beta\cdot\mathbf{n} = 1-\beta\cos\theta$, the second result follows.

(iv) [2 marks] In the non-relativistic limit, the equation above becomes

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\dot{u}^2\sin^2\theta,$$

as required.

(v) [6 marks] The angular dependence is in the factors

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5},$$

and so the maximum intensity occurs in the direction where this has a maximum, given by simple calculus as where $2\sin\theta\cos\theta(1-\beta\cos\theta) = 5\sin^2\theta\,\beta\sin\theta$, which gives a quadratic equation for $\cos\theta$ -

$$3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0.$$

There is only one root with a real value for θ, and that is the one given.