ELECTROMAGNETIC THEORY

SUPPLEMENTARY NOTES

These notes are intended to clarify and amplify some of the earlier notes on scattering theory.

Let us start from Maxwell's equations, in their form appropriate to fields in a medium. The *homogeneous* equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

remain unchanged, and allow us to introduce the potentials $\mathbf{A}$ and Φ as before, in terms of which we still have

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\dot{\mathbf{A}} - \nabla\Phi.$$

The *inhomogeneous* equations

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}}$$

introduce the fields $\mathbf{D}$ and $\mathbf{H}$ which are essentially phenomenological, macroscopic fields. They are related to the more fundamental fields $\mathbf{E}$ and $\mathbf{B}$ by the *constitutive relations*:

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}.$$

(In a conductor we also have $\mathbf{j} = \sigma\mathbf{E}$.) Although we have written the permittivity ϵ, the permeability μ and the conductivity σ as scalar quantities, they are more generally tensors in anisotropic media. We will for the present suppose them to be constants. It is also useful to introduce the *polarisation* $\mathbf{P}$ and the *magnetisation* $\mathbf{M}$ by

$$\mathbf{D} - \epsilon_0\mathbf{E} = \mathbf{P}, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}).$$

The relative permittivity ϵ_r and the relative permeability μ_r are defined using the vacuum constants ϵ_0, μ_0 by $\epsilon = \epsilon_0\epsilon_r, \quad \mu = \mu_0\mu_r$.

Direct substitution allows one to conclude that

$$\Big(\epsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\Phi = \frac{\rho}{\epsilon},$$
$$\Big(\epsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\mathbf{A} = \mu\mathbf{j},$$

where we have used the freedom to choose a gauge in which

$$\nabla \cdot \mathbf{A} + \epsilon\mu\dot{\Phi} = 0.$$

In the absence of sources, $\mathbf{j} = 0 = \rho$, these have solutions of the form $\exp[-i(\omega t - \mathbf{k}\cdot\mathbf{x})]$ of propagating waves, with phase velocity $v = \omega/k$ given by

$$v^2 = \frac{1}{\epsilon\mu}.$$

Of course, in vacuum we have

$$c^2 = \frac{1}{\epsilon_0\mu_0}.$$

Note that when there are no free charges ($\rho = 0$), we may take $\Phi = 0$, so that we are in *radiation gauge*, when also $\nabla \cdot \mathbf{A} = 0$. Then we have $\mathbf{E} = -\dot{\mathbf{A}},\ \mathbf{B} = \nabla \times \mathbf{A}$. For the plane waves, we can even simplify this to give $\mathbf{E} = i\omega\mathbf{A},\ \mathbf{B} = i\mathbf{k} \times \mathbf{A}$, and using the gauge condition $\nabla \cdot \mathbf{A} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{A} = 0$, we have illustrated the fact that $\mathbf{k}, \mathbf{E}, \mathbf{B}$ are mutually orthogonal vectors: electromagnetic waves are *transverse*.

In the absence of sources, the $\mathbf{E}$ and $\mathbf{B}$ fields satisfy wave equations

$$\nabla^2\mathbf{B} = \frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2},$$
$$\nabla^2\mathbf{E} = \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

In conductors, the inclusion of $\mathbf{j} = \sigma\mathbf{E}$ modifies these equations to give

$$\nabla^2\mathbf{B} = \frac{1}{v^2}\frac{\partial^2\mathbf{B}}{\partial t^2} + \sigma\mu\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla^2\mathbf{E} = \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \sigma\mu\frac{\partial\mathbf{E}}{\partial t}.$$

For waves of frequency ω, the effect of this is to replace $v^{-2} = \epsilon\mu$ by $\epsilon\mu + i\sigma\mu/\omega$, so in effect to add an imaginary part σ/ω to the permittivity.

We will also wish later to make use of the equations

$$\Box\mathbf{A} = \mu_0(\mathbf{j} + \nabla\times\mathbf{M} + \dot{\mathbf{P}}) - \nabla(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\dot{\Phi})$$
$$\Box\Phi = c^2\mu_0(\rho - \nabla\cdot\mathbf{P}) - \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\dot{\Phi}),$$

or, on imposing the Lorentz condition $\partial_\mu A^\mu = 0$

$$\Box\mathbf{A} = \mu_0(\mathbf{j} + \nabla\times\mathbf{M} + \dot{\mathbf{P}})$$
$$\Box\Phi = c^2\mu_0(\rho - \nabla\cdot\mathbf{P}).$$

We have considered extensively the case when all the dynamic quantities, fields and sources alike, oscillate with the same frequency, so that their time dependence may be given by a factor $e^{-i\omega t}$. We are thereby lead to consider equations of the general form

$$\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = f(\mathbf{x}),$$

where $k = v\omega$ for waves propagating with velocity v in a medium, or $k = c\omega$ in vacuum. In the absence of the source term $f(\mathbf{x})$, the solutions will be superpositions of $\psi = \exp[i\mathbf{k}\cdot\mathbf{x}]$, or restoring the time-dependence, $\psi(\mathbf{x}, t) = \exp[-i(\omega t - \mathbf{k}\cdot\mathbf{x})]$, a plane wave travelling in the direction of the vector $\mathbf{k}$. Solution in the presence of the source term is handled by the method of Green's functions: the solution for the particular case

$$(\nabla^2 + k^2)G_k(\mathbf{x}, \mathbf{y}) = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

which is (using "outgoing" boundary conditions)

$$G_k(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}.$$

We then have as a solution

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int d^3y\, G_k(\mathbf{x}, \mathbf{y})\, f(\mathbf{y}),$$

to which can be added if appropriate a solution of the homogeneous equation.

In this way we may obtain the potentials

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3y\, \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\mathbf{j}(\mathbf{y}),$$
$$\Phi(\mathbf{x}) = \frac{\epsilon_0}{4\pi}\int d^3y\, \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y})$$

for *given* oscillating charge and current source densities. These equations are equivalent to

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int d^3y\, j^\mu(\mathbf{y}, t_{\text{ret}})\frac{1}{|\mathbf{x}-\mathbf{y}|}.$$

When the field point $\mathbf{x}$ is far from the sources, which we may take to be localised in the vicinity of the origin, these are well approximated by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}\mathbf{j}(\mathbf{y}),$$
$$\Phi(\mathbf{x}) = \frac{\epsilon_0}{4\pi}\frac{e^{ikr}}{r}\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}\rho(\mathbf{y}).$$

Here $\mathbf{k}$ is a vector of magnitude k in the direction of $\mathbf{x}$, and $r = |\mathbf{x}|$; the solutions have as their r-dependence just the characteristic outgoing spherical wave e^{ikr}/r, modulated in direction through the dependence on $\mathbf{k}$. In deriving the electric field from $\mathbf{E} = -\dot{\mathbf{A}} - \boldsymbol{\nabla}\Phi = ikc\mathbf{A} - \boldsymbol{\nabla}\Phi$, the role of the Φ contribution is to cancel off the component of $\mathbf{E}$ in the direction of $\mathbf{k}$. We then find that the radiated fields are given by

$$\mathbf{B} = i\mathbf{k}\times\mathbf{A}, \qquad \mathbf{E} = ikc\mathbf{A}_T,$$

(the suffix T denoting transverse).

The *multipole* approximation to the fields is derived from a systematic expansion of the Green function $G_k(\mathbf{x},\mathbf{y})$, which leads off with

$$\frac{e^{ikr}}{r}\Big(1 - i\mathbf{k}\cdot\mathbf{y}\big(1 - \frac{1}{ikr}\big)\Big).$$

This expansion then expresses $\mathbf{A}$ in terms of succesive multipoles of the source, electric and magnetic. The monopole term (electric only, there are no magnetic monopoles!) makes no contribution to the radiation. The leading contributions will therefore usually be the dipole terms

$$\mathbf{A}(\mathbf{x}) = -ick\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\Big(\mathbf{p} - \frac{\mathbf{k}\times\mathbf{m}}{kc}\Big),$$

the potential produced by oscillating electric and magnetic dipoles $\mathbf{p}$, $\mathbf{m}$ respectively. The corresponding electric and magnetic fields are thus

$$\mathbf{E}(\mathbf{x}) = \frac{k^2}{4\pi\epsilon_0}\Big(\mathbf{p}_T - \frac{\mathbf{k}\times\mathbf{m}}{kc}\Big)\frac{e^{ikr}}{r}$$
$$\mathbf{B}(\mathbf{x}) = \frac{k^2\mu_0}{4\pi}\Big(\mathbf{m}_T + \frac{c\mathbf{k}\times\mathbf{p}}{k}\Big)\frac{e^{ikr}}{r} = \frac{1}{ck}\mathbf{k}\times\mathbf{E}.$$

Similar methods apply to the discussion of *scattering*, when the radiation is not generated by *given* sources $\rho, \mathbf{j}$, but by the electric and magnetic polarisation ($\mathbf{P}$ and $\mathbf{M}$ respectively) of the scatterer *induced* by the incident fields. So we now have instead of $\Box\mathbf{A} = \mu_0\mathbf{j}$ the corresponding equation

$$\Box\mathbf{A} = \mu_0(\boldsymbol{\nabla}\times\mathbf{M} + \dot{\mathbf{P}}).$$

This time it is appropriate to add a solution to the homogeneous equation, representing the incident radiation, so as to obtain

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_{\text{in}}(\mathbf{x}) + \frac{\mu_0}{4\pi}\int d^3y\, G_k(\mathbf{x},\mathbf{y})(\boldsymbol{\nabla}\times\mathbf{M} - i\omega\mathbf{P})(\mathbf{y}).$$

If the incident wave is taken to be a plane wave in the direction $\mathbf{k}_0$ with polarisation $\boldsymbol{\epsilon}_0$ and amplitude A_0, we have

$$\mathbf{A}_{\text{in}}(\mathbf{x}) = A_0\boldsymbol{\epsilon}_0 e^{i\mathbf{k}_0\cdot\mathbf{x}}$$

and the scattered wave becomes, at large distances,

$$\mathbf{A}_{\text{SC}}(\mathbf{x}) = \frac{e^{ikr}}{r}\frac{\mu_0}{4\pi}\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\times\mathbf{M} - ick\mathbf{P})(\mathbf{y})$$
$$= A_0\frac{e^{ikr}}{r}\mathbf{F}$$

Again the effect of the scalar potential contribution to the fields is just to cancel off the longitudinal component of $\mathbf{E}$; the corresponding electric and magnetic fields are transverse, that is to say perpendicular to $\mathbf{k}$. We have introduced

$$\begin{aligned}\mathbf{F} &= \frac{1}{A_0}\frac{\mu_0}{4\pi}\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\times\mathbf{M} - ick\mathbf{P})(\mathbf{y}) \\ &= \frac{1}{A_0}\frac{\mu_0}{4\pi}(-ick)\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}\Big(\mathbf{P} - \frac{\mathbf{k}\times\mathbf{M}}{kc}\Big)(\mathbf{y}),\end{aligned}$$

which depends implicitly on the wave-vector $\mathbf{k}_0$ and polarisation $\boldsymbol{\epsilon}_0$ of the incident radiation, and explicitly on the wave-vector $\mathbf{k}$ of the outgoing scattered radiation. The differential cross-section for scattered radiation with polarisation $\boldsymbol{\epsilon}$ is then given by

$$\frac{d\sigma}{d\Omega} = |F(\mathbf{k},\boldsymbol{\epsilon};\mathbf{k}_0,\boldsymbol{\epsilon}_0)|^2,$$

with

$$F = \boldsymbol{\epsilon}^*\cdot\mathbf{F}.$$

F is called the *scattering amplitude.*

The *optical theorem* relates the imaginary part of the forward scattering amplitude to the total cross-section. The total cross-section is the rate at which energy is removed from the incident wave divided by the flux of energy in the incident wave. With $\mathbf{A}_{\text{in}}$ as given, since we have $\mathbf{E}_0 = ick\mathbf{A}_{\text{in}} = ickA_0\boldsymbol{\epsilon}_0 \exp i\mathbf{k}_0\cdot\mathbf{x}$ and $\mathbf{B}_0 = i\mathbf{k}_0\times\mathbf{A}_{\text{in}} = iA_0\mathbf{k}_0\times\boldsymbol{\epsilon}_0\exp i\mathbf{k}_0\cdot\mathbf{x}$, the (time-averaged) incident flux is $ck^2|A_0|^2/2\mu_0$. The rate of flow of energy into the scatterer is given by integrating the Poynting vector across any surface S surrounding the scatterer as

$$P_{\text{abs}} = -\frac{1}{2\mu_0}\int_S \mathbf{n}\cdot\Re[\mathbf{E}\times\mathbf{B}^*]\,dS,$$

where $\mathbf{n}$ is the outwards normal so that there is a minus sign for the *inwards* flow of energy, and the factor of a half is for time-averaging. In addition to this absorbed power, there is power scattered out of the incident beam, given by

$$P_{\text{scattd}} = \frac{1}{2\mu_0}\int_S \mathbf{n}\cdot\Re[\mathbf{E}_s\times\mathbf{B}_s^*]\,dS.$$

With $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_s,\ \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_s$ this gives

$$\begin{aligned}P &= P_{\text{abs}} + P_{\text{scattd}} \\ &= -\frac{1}{2\mu_0}\int_S \mathbf{n}\cdot\Re[\mathbf{E}_s\times\mathbf{B}_0^* + \mathbf{E}_0^*\times\mathbf{B}_s]\,dS \\ &= \frac{1}{2\mu_0}\int_S \mathbf{n}\cdot\Re[iA_0^*e^{-i\mathbf{k}_0\cdot\mathbf{y}}(\mathbf{E}_s\times(\mathbf{k}_0\times\boldsymbol{\epsilon}_0^*) + ck\boldsymbol{\epsilon}_0^*\times\mathbf{B}_s)]\,dS \\ &= \frac{1}{2\mu_0}\Im\Big\{A_0^*\boldsymbol{\epsilon}_0^*\cdot\int_S e^{-i\mathbf{k}_0\cdot\mathbf{y}}[ck\mathbf{n}\times\mathbf{B}_s - (\mathbf{n}\times\mathbf{E}_s)\times\mathbf{k}_0]\,dS\Big\}.\end{aligned}$$

Hence we have for the total cross-section σ_{tot}, which is P divided by the incident flux,

$$\sigma_{\text{tot}} = \frac{1}{ck^2}\Im\Big\{A_0^{-1}\boldsymbol{\epsilon}_0^*\cdot\int_S e^{-i\mathbf{k}_0\cdot\mathbf{y}}[ck\mathbf{n}\times\mathbf{B}_s - (\mathbf{n}\times\mathbf{E}_s)\times\mathbf{k}_0]\,dS\Big\}.$$

Now the scattering amplitude is

$$F = \frac{1}{A_0}\frac{\mu_0}{4\pi}\boldsymbol{\epsilon}^*\cdot\int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\times\mathbf{M} + \dot{\mathbf{P}})(\mathbf{y}),$$

and the integral in this expression, which is taken over any volume enclosing the scatterer can be expressed by a succession of vector identities as follows:

$$
\begin{aligned}
\mu_0 \int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\times\mathbf{M}+\dot{\mathbf{P}})(\mathbf{y}) \\
&= \int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}\Box\mathbf{A}(\mathbf{y}) \\
&= \int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}\Box\mathbf{A}_s(\mathbf{y}) \qquad (\Box\mathbf{A}_{\text{in}}=0) \\
&= \int d^3y\, e^{-i\mathbf{k}\cdot\mathbf{y}}(-k^2-\nabla^2)\mathbf{A}_s(\mathbf{y}) \qquad\qquad = \textstyle\int d^3y\,[\mathbf{A}_s\nabla^2 e^{-i\mathbf{k}\cdot\mathbf{y}} - e^{-i\mathbf{k}\cdot\mathbf{y}}\nabla^2\mathbf{A}_s] \\
&= \int_S [\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} - e^{-i\mathbf{k}\cdot\mathbf{y}}\,(\mathbf{n}\cdot\boldsymbol{\nabla})\mathbf{A}_s]\,dS \qquad \text{(Green's theorem)} \\
&= \int_S [2\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} - (\mathbf{n}\cdot\boldsymbol{\nabla})(e^{-i\mathbf{k}\cdot\mathbf{y}}\mathbf{A}_s)]\,dS \\
&= \int_S 2\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS - \int d^3y\,\nabla^2(\mathbf{A}_s e^{-i\mathbf{k}\cdot\mathbf{y}}) \qquad \text{(divergence theorem)} \\
&= \int_S 2\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS - \int d^3y\,[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(\mathbf{A}_s e^{-i\mathbf{k}\cdot\mathbf{y}})) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\mathbf{A}_s e^{-i\mathbf{k}\cdot\mathbf{y}}))] \\
&= \int_S [2\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} - \mathbf{n}\,\boldsymbol{\nabla}\cdot(\mathbf{A}_s e^{-i\mathbf{k}\cdot\mathbf{y}}) + \mathbf{n}\times(\boldsymbol{\nabla}\times(\mathbf{A}_s e^{-i\mathbf{k}\cdot\mathbf{y}}))]\,dS \qquad \text{(Green again)} \\
&= \int_S [2\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} - \mathbf{n}\,e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\cdot\mathbf{A}_s) - \mathbf{n}\,(\mathbf{A}_s\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} + e^{-i\mathbf{k}\cdot\mathbf{y}}\mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{A}_s) \\
&\qquad - \mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} + (\mathbf{n}\cdot\mathbf{A}_s)\,\boldsymbol{\nabla}e^{-i\mathbf{k}\cdot\mathbf{y}}]\,dS \\
&= \int_S [\mathbf{A}_s\,(\mathbf{n}\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}} + e^{-i\mathbf{k}\cdot\mathbf{y}}\mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{A}_s) + (\mathbf{n}\cdot\mathbf{A}_s)\,\boldsymbol{\nabla}e^{-i\mathbf{k}\cdot\mathbf{y}} - \mathbf{n}\,e^{-i\mathbf{k}\cdot\mathbf{y}}(\boldsymbol{\nabla}\cdot\mathbf{A}_s) - \mathbf{n}\,(\mathbf{A}_s\cdot\boldsymbol{\nabla})e^{-i\mathbf{k}\cdot\mathbf{y}}]\,dS \\
&= \int_S [-i\mathbf{A}_s\,(\mathbf{n}\cdot\mathbf{k}) + \mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{A}_s) - i\mathbf{k}\,(\mathbf{n}\cdot\mathbf{A}_s) - \mathbf{n}\,(\boldsymbol{\nabla}\cdot\mathbf{A}_s) + i\mathbf{n}\,(\mathbf{k}\cdot\mathbf{A}_s)]e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS.
\end{aligned}
$$

We may now impose the radiation gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, and use

$$
ikc\mathbf{A}_s = \mathbf{E}_s, \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{A}_s = \mathbf{B}_s
$$

as well as the fact that the scattered radiation is in the direction of $\mathbf{n}$ and is transverse so that $\mathbf{n}\cdot\mathbf{E}_s=0$ to write this as

$$
\frac{1}{kc}\int_S [-\mathbf{E}_s\,(\mathbf{n}\cdot\mathbf{k}) + kc\mathbf{n}\times\mathbf{B}_s + \mathbf{n}\,(\mathbf{k}\cdot\mathbf{E}_s)]e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS.
$$

This then gives for the scattering amplitude

$$
\begin{aligned}
F(\mathbf{k},\boldsymbol{\epsilon};\mathbf{k}_0,\boldsymbol{\epsilon}_0) &= \frac{1}{4\pi kcA_0}\boldsymbol{\epsilon}^*\cdot\int_S [-\mathbf{E}_s\,(\mathbf{n}\cdot\mathbf{k}) + kc\mathbf{n}\times\mathbf{B}_s + \mathbf{n}\,(\mathbf{k}\cdot\mathbf{E}_s)]e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS \\
&= \frac{1}{4\pi kcA_0}\boldsymbol{\epsilon}^*\cdot\int_S [kc\mathbf{n}\times\mathbf{B}_s - (\mathbf{n}\times\mathbf{E}_s)\times\mathbf{k}]e^{-i\mathbf{k}\cdot\mathbf{y}}\,dS.
\end{aligned}
$$

Comparing this with the previously obtained expression for the total cross-section, we arrive at

$$
\sigma_{\text{tot}} = \frac{4\pi}{k}\Im F(\mathbf{k}_0,\boldsymbol{\epsilon}_0;\mathbf{k}_0,\boldsymbol{\epsilon}_0),
$$

which is the *optical theorem*, expressing the total cross-section as $4\pi/k$ times the imaginary part of the forward scattering amplitude.