Queen Mary University of London

BSc/MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: 12^{th} May, 2010

Time: 10.00 - 12.30

Instructions: **Answer ONLY THREE questions. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper. Course work comprises 10 % of the final mark.**

Numeric calculators are not permitted in this examination. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. S. Ramgoolam , Dr. D. Berman

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Question 1

(a) The frequency-dependent relative permittivity $\epsilon_r(\omega)$ is defined by

$$\mathbf{D}(\mathbf{x},\omega) = \epsilon_0\epsilon_r(\omega)\mathbf{E}(\mathbf{x},\omega).$$

Hence, show that

$$\mathbf{D}(\mathbf{x},t) = \epsilon_0\mathbf{E}(\mathbf{x},t) + \epsilon_0\int_{-\infty}^{\infty} G(\tau)\mathbf{E}(\mathbf{x},t-\tau)d\tau$$

where

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\epsilon_r(\omega)-1)e^{-i\omega\tau}d\omega.$$

[8]

In a simple molecular model for the permittivity

$$\epsilon_r(\omega) - 1 = \frac{\omega_P^2}{(\omega + \frac{i\gamma}{2} - \omega_R)(\omega + \frac{i\gamma}{2} + \omega_R)}$$

for real ω_P, ω_R. Using contour integration

(b) Explain why $\mathbf{D}(\mathbf{x},t)$ is causally related to $\mathbf{E}(\mathbf{x},t)$. **[5]**

(c) Show that for $\tau > 0$

$$G(\tau) = \omega_P^2 e^{\frac{-\gamma\tau}{2}}\frac{sin(\omega_R\tau)}{\omega_R}$$

[7]

Question 2

Consider the Maxwell equations in a vacuum with sources -

(a) Show that the first two of these equations may be solved by introducing the potentials $\mathbf{A}$ and Φ, and writing

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Show that the other two Maxwell equations then become

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho,$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -\mu_0\mathbf{J}.$$

[6]

(b) Using $A^\mu = (A^0 = \frac{\Phi}{c}, A^i)$ and the definition $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ show that

$$F^{ij} = -\epsilon^{ijk}B^k$$
$$\frac{E^i}{c} = -F^{0i}$$

[4]

(c) Prove

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0$$

and show that taking one index to be 0 and two indices to be spatial gives one of Maxwell's equations. **[5]**

(d) Show that two of the remaining Maxwell's equations are equivalent to $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$. where j^β is related to charge density ρ and current density $\mathbf{J}$ by $j^\alpha = (\rho c, J^i)$ **[5]**

Question 3

The Lagrangian for electromagnetic fields, in the absence of sources, is

$$L = \frac{-1}{4\mu_0} \int d^3x F^{\alpha\beta} F_{\alpha\beta}$$

In the following, you may freely use the equation of motion $\partial_\mu F^{\mu\alpha} = 0$ which follows from this Lagrangian.

(a) The canonical stress tensor is expressed in terms of the Lagrangian density $\mathcal{L}$ by

$$T^{\nu}_{\ \mu} = \partial_\mu A_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)} - \delta^\nu_\mu \mathcal{L}$$

Derive an expression for $T^{\nu}_{\ \mu}$ in terms of the tensors F and A. **[4]**

(b) Show that it satisfies $\partial_\mu T^{\mu}_{\ \nu} = 0$. You may assume the equation of motion $\partial_\mu F^{\mu\alpha} = 0$ in the absence of sources. **[4]**

(c) Show that $T^{\mu\nu}$ differs from the symmetric stress tensor

$$\Theta^{\mu\nu} = \frac{-1}{\mu_0}[F^{\lambda\mu} F_{\lambda}^{\ \nu} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}]$$

by a total derivative. **[4]**

(d) Show that $\partial_\mu \Theta^{\mu\nu} = j^\lambda F^\nu_\lambda$ in the presence of sources. You may use the fact that two of Maxwell's equations can be expressed as $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$. **[4]**

(e) By computing Θ^{00}, Θ^{0i}, show how the sourceless equation $\partial_\mu \Theta^{\mu\nu} = 0$ can be used to express the conservation of energy. You can assume that $F^{0i} = -\frac{E^i}{c}, F^{ij} = -\epsilon^{ijk} B^k$ **[4]**

Question 4

The electric field at large distances from a charge q following a trajectory $\mathbf{r} = \mathbf{r}(t)$, with instantaneous velocity $\mathbf{u} = \frac{d\mathbf{r}}{dt} = c\boldsymbol{\beta}$, $|\boldsymbol{\beta}| << 1$ is given by

$$\mathbf{E_{far}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{1}{R}(\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}}))\right]_{\rm ret}$$

(a) Explain the meaning of the notation $[\ldots]_{\rm ret}$, and define the distance R and the direction vector $\mathbf{n}$. **[5]**

(b) Assuming that the corresponding magnetic field is given by

$$\mathbf{B_{far}} = \left[\mathbf{n} \times \mathbf{E_{far}}\right]_{\mathbf{ret}}/c,$$

show that at large distances, the Poynting energy-flux vector is

$$\mathbf{S_{far}} = \frac{1}{\mu_o c}|\mathbf{E_{far}}|^2\mathbf{n}$$

[5]

(c) Derive the Larmor formula

$$P = \frac{q^2}{6\pi\epsilon_0 m^2 c^3}\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}$$

for the total instantaneous power radiated by a non-relativistic accelerated charge. **[5]**

(d) Write down the Lienard formula which is a relativistic generalization of the Larmor formula. Apply it to relativistic motion in a circle radius ρ with angular frequency ω, under conditions where the rate of energy loss per revolution is small, to derive the formula for the power loss

$$P = \frac{q^2}{6\pi\epsilon_0}\frac{c\beta^4\gamma^4}{\rho^2}$$

You can assume that $|\frac{d\mathbf{p}}{dt}| = \omega|\mathbf{p}|$. **[5]**

Question 5

(a) Derive, for the Dirac delta function, the equation

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|\frac{\partial f}{\partial x}(x_0)|}$$

for the case where $f(x)$ has a single simple zero on the real axis at $x = x_0$. **[4]**

(b) Using the formula $d\tau^2 = dt^2 - c^{-2} d\mathbf{x} \cdot d\mathbf{x}$ for the infinitesimal change in the proper time of a particle, derive expressions for the components U^0, U^i of the four-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ in terms of $\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}}}$ and $\mathbf{u} = \boldsymbol{\beta} c$. **[3]**

(c) The scalar potential due to a moving charged particle, along a trajectory $\mathbf{r}(t')$, is given by

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0} \int dt' \frac{\delta(t - t' - \frac{|\mathbf{x} - \mathbf{r}(t')|}{c})}{|\mathbf{x} - \mathbf{r}(t')|}.$$

Show that

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{r}(t_r)| - \boldsymbol{\beta}(t_r) \cdot (\mathbf{x} - \mathbf{r}(t_r))}$$

where $c(t - t_r) = |\mathbf{x} - \mathbf{r}(t_r)|$. **[6]**

(d) Show that the above equation for Φ is the time-component of a 4-vector equation

$$A^\mu = \frac{\mu_0 q c}{4\pi} \frac{U^\mu}{(x^\nu - r^\nu) U_\nu}.$$

[5]

(e) Write down a formula for the vector potential of a moving charge. **[2]**

Formula Sheet

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \\
\nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}, \\
\nabla \times (\psi \mathbf{a}) &= (\nabla \psi) \times \mathbf{a} + \psi (\nabla \times \mathbf{a}), \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \\
\nabla \big(\psi(r)\big) &= \mathbf{n} \psi'(r).
\end{aligned}$$

Maxwell's equations:

$$\begin{aligned}
\nabla \cdot \mathbf{B} = 0, \qquad & \nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t}; \\
\nabla \cdot \mathbf{D} = \rho, \qquad & \nabla \times \mathbf{H} = \mathbf{J} + \tfrac{\partial \mathbf{D}}{\partial t}.
\end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

$$\nabla \cdot \mathbf{J} + \dot{\rho} = 0.$$

For Fourier transforms :

$$\begin{aligned}
f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega \\
f(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \\
\frac{1}{2\pi} \int dt e^{i\omega(t-t')} &= \delta(t - t')
\end{aligned}$$

For linear isotropic media:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}.$$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dx^\alpha \eta_{\alpha\beta} dx^\beta.$$

$$\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0 \\ -1 & \text{if } \alpha = \beta = 1, 2, 3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right), \qquad \partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right).$$

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial^\beta \partial_\alpha A^\alpha = \mu_0 j^\beta; \qquad F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

Energy density and Poynting vector :

$$\begin{aligned}
u &= \frac{\epsilon_0}{2} (\mathbf{E}^2 + c^2 \mathbf{B}^2) \\
\vec{\mathbf{S}} &= \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})
\end{aligned}$$