MSci 4261 ELECTROMAGNETISM

LECTURE NOTES VIII

In these notes we will formulate classical electrodynamics in an explicitly Lorentz covariant manner.

8.1 The Charge-Current Density Four-Vector

Consider an element of charge at rest in a frame K, occupying the volume element $dx^1\,dx^2\,dx^3$, so that

$$q = \rho\, dx^1\, dx^2\, dx^3.$$

In the frame K' this same element of charge will occupy the volume element $dx'^1\, dx'^2\, dx'^3 = (\gamma^{-1} dx^1)\, dx^2\, dx^3$ and since the total amount of charge in the volume element is the same in both frames,

$$q = \rho'\, dx'^1\, dx'^2\, dx'^3,$$

and it follows that

$$\rho' = \gamma\rho.$$

But note also that in K' the element of charge is *moving*, with a velocity $\mathbf{u}' = -\boldsymbol{\beta}c$ so that there is also a current density in K' given by

$$\mathbf{j}' = \rho'\mathbf{u}' = -\gamma\boldsymbol{\beta}c\rho.$$

So we have $(\rho; \mathbf{j} = 0)$ in K transforming to $(\rho' = \gamma\rho; \mathbf{J}' = -\gamma\boldsymbol{\beta}c\rho)$ in K', which suggests writing $j^0 = c\rho, \mathbf{J} = (J^1, J^2, J^3)$ since then the transformation of j^μ may be recognised as that of a four-vector. In fact our element of charge has a four-velocity U^μ, and we have

$$j^\mu = \rho_0 U^\mu$$

where ρ_0 is the charge density in the frame in which the element of charge is at rest. We may conclude that the charge-current density $j = (c\rho, \mathbf{J})$ is a contravariant 4-vector.

Note also that the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

is

$$\frac{\partial j^0}{\partial x^0} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^2}{\partial x^2} + \frac{\partial j^3}{\partial x^3} = 0$$

or

$$\partial_\mu j^\mu = 0$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

The continuity equation is thus a manifestly covariant equation.

8.2 The Lorentz Force

Having found the transformation rule for the *sources* j^μ it is natural to turn to the question of the transformation rules for the fields themselves. As a step in this direction, recall that the fields are in fact *defined* through the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

on a test particle of charge q moving with velocity $\mathbf{u}$. Since $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, we have

$$\begin{aligned}\frac{d\mathbf{p}}{d\tau} &= q(\mathbf{E} + \mathbf{u} \times \mathbf{B})\frac{dt}{d\tau} \\ &= q(\mathbf{E} + \mathbf{u} \times \mathbf{B})\gamma_u \\ &= q(\frac{1}{c}\mathbf{E}U^0 + \mathbf{U} \times \mathbf{B}),\end{aligned}$$

and also the rate of change of the energy $\mathcal{E}$ of the test particle is given by

$$\begin{aligned}\frac{dp^0}{d\tau} &= \frac{1}{c}\frac{dt}{d\tau}\frac{d\mathcal{E}}{dt} \\ &= \frac{1}{c}\gamma q \mathbf{E} \cdot \mathbf{u} \\ &= \frac{q}{c}\mathbf{E} \cdot \mathbf{U}.\end{aligned}$$

Thus

$$\frac{d}{d\tau}\begin{pmatrix} p^0 \\ \mathbf{p} \end{pmatrix} = \frac{q}{c}\begin{pmatrix} \mathbf{E} \cdot \mathbf{U} \\ \mathbf{E}U^0 + c\mathbf{U} \times \mathbf{B} \end{pmatrix},$$

which may be written as

$$\frac{d}{d\tau}p^\alpha = qF^{\alpha\beta}U_\beta$$

where $F^{\alpha\beta}$ is a certain second rank tensor, the components of which are given in terms of the $\mathbf{E}$ and $\mathbf{B}$ fields. It is possible to obtain this result directly, but it is easier and also more useful to proceed by introducing the *potential four-vector.*

8.3 The Potential Four-Vector

Recall that for any *potentials* $\mathbf{A}, \Phi$ with the definitions

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}, \\ \mathbf{E} &= -\boldsymbol{\nabla}\Phi - \dot{\mathbf{A}},\end{aligned}$$

two of Maxwell's equations, namely the pair

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{E} + \dot{\mathbf{B}} &= 0,\end{aligned}$$

are automatically satisfied. Furthermore, the gauge transformations

$$\begin{aligned}\mathbf{A} &\to \mathbf{A} + \boldsymbol{\nabla}\chi \\ \Phi &\to \Phi - \dot{\chi}\end{aligned}$$

to the vector and scalar potentials leave the fields $\mathbf{E}$ and $\mathbf{B}$ unaltered - the latter are are *gauge invariant.*

Under a gauge transformation, the quantity

$$L \equiv \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\dot{\Phi}$$

transforms as

$$L \to L' = L - \Box\chi,$$

using the *D'Alembertian* $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ (since

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial^\alpha\partial_\alpha,$$

it is a Lorentz scalar). By suitable choice of χ it is always possible to impose the Lorentz gauge

$$L = \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\dot{\Phi} = 0.$$

This condition is indeed Lorentz invariant: if true in one frame, it remains satisfied in any other.

Whatever the choice of gauge, as we have seen, the pair of Maxwell's equations which do not involve the source terms are automatically satisfied. We now turn to the other pair of equations, namely

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \mu_0 c^2 \rho$$
$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \dot{\mathbf{E}}).$$

Written in terms of the potentials, and making use of the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A},$$

these become

$$\Box \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\dot{\Phi}) = \mu_0 \mathbf{J}$$

and

$$\Box \frac{1}{c}\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\dot{\Phi}) = \mu_0 c \rho.$$

Recalling that we have defined

$$L \equiv \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\dot{\Phi},$$

and suppose that we define

$$A^0 \equiv \frac{1}{c}\Phi,$$

what we have is

$$\Box A^\mu - \eta^{\mu\nu}\partial_\nu L = \mu_0 j^\mu.$$

And since we know that j^μ is a four-vector, this suggests that A^μ is likewise a four-vector, in which case L is indeed a Lorentz scalar, and the field equations satisfied by the potentials A^μ are seen to be Lorentz covariant.

Note that if we impose the Lorentz gauge condition, in which case $L = 0$, the equations simplify to

$$\Box A^\mu = \mu_0 j^\mu; \qquad \partial_\mu A^\mu = 0.$$

8.4 The Field-Strength Tensor

We have expressed the electric and magnetic fields in terms of the potentials, and we know how the potentials change under a Lorentz transformation,

$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu.$$

From this follows the transformation laws for the electric and magnetic fields. To obtain them most simply, note that the field components are all of them of the form

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha,$$

where $\partial^\alpha \equiv \eta^{\alpha\beta}\partial_\beta \Rightarrow \partial^0 = \partial_0;\ \partial^1 = -\partial_1,$ *etc.* For example, $B^x = \frac{\partial A^z}{\partial y} - \frac{\partial A^y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$ and $\frac{1}{c}E^x = -\frac{1}{c}\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A^x}{\partial t} = -(\partial^0 A^1 - \partial^1 A^0)$. Thus the components of the *tensor* $F^{\alpha\beta}$ arranged as a matrix are

$$\| F^{\alpha\beta} \| = \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & -B^z & B^y \\ E^y/c & B^z & 0 & -B^x \\ E^z/c & -B^y & B^x & 0 \end{pmatrix}.$$

This second-rank contravariant antisymmetrical tensor is the *field-strength tensor*; because of its antisymmetry $F^{\alpha\beta} = -F^{\beta\alpha}$, it has only six independent components. We can of course lower its indices to obtain $F_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}F^{\mu\nu}$, with

$$\| F_{\alpha\beta} \| = \begin{pmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & -B^z & B^y \\ -E^y/c & B^z & 0 & -B^x \\ -E^z/c & -B^y & B^x & 0 \end{pmatrix}.$$

For any choice of potentials A_μ, the fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ automatically satisfy

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0,$$

and expressed in terms of the electric and magnetic fields, these equations are once again the homogeneous pair of Maxwell's equations, *i.e.,* those which do not involve the sources.

As for the inhomogeneous pair of Maxwell's equations, these are obtained from

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \\ &= \Box A^\beta - \partial^\beta(\partial_\alpha A^\alpha) \\ &= \Box A^\beta - \partial^\beta L \\ &= \mu_0 j^\beta. \end{aligned}$$

Returning for a moment to the force equation, which we had written as

$$\frac{d}{d\tau}\begin{pmatrix} p^0 \\ \mathbf{p} \end{pmatrix} = q\begin{pmatrix} \frac{1}{c}\mathbf{E}\cdot\mathbf{U} \\ \frac{1}{c}\mathbf{E}U^0 + \mathbf{U}\times\mathbf{B} \end{pmatrix},$$

we now find that

$$\begin{aligned} \frac{d}{d\tau}p^0 &= q(-F^{01}U^1 - F^{02}U^2 - F^{03}U^3) \\ &= q(F^{01}U_1 + F^{02}U_2 + F^{03}U_3) \\ &= qF^{0\beta}U_\beta, \end{aligned}$$

and for example

$$\begin{aligned} \frac{d}{d\tau}p^1 &= q(-F^{01}U^0 + U^2(-F^{12}) - U^3F^{13}) \\ &= q(F^{10}U_0 + F^{12}U_2 + F^{13}U_3) \\ &= qF^{1\beta}U_\beta. \end{aligned}$$

So as advertised,

$$\frac{d}{d\tau}p^\alpha = qF^{\alpha\beta}U_\beta.$$

To spell out what precisely *is* the Lorentz transformation law for the electromagnetic field, we recall that the transformation law for any contravariant tensor such as $F^{\mu\nu}$ is given by

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda F^{\kappa\lambda}$$

where for the standard boost the coefficients $\Lambda^\mu{}_\kappa$ may be arranged as the matrix

$$\| \Lambda^\mu{}_\kappa \| = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So in terms of *matrices*, we have

$$F \to F' = \Lambda F \Lambda^T,$$

(Λ^T denoting the matrix transpose of Λ). Straightforward matrix multiplication then yields

$$\begin{aligned}
E'^x &= E^x \\
E'^y &= E^y \cosh\zeta - B^z\, c \sinh\zeta \\
E'^z &= E^z \cosh\zeta + B^y\, c \sinh\zeta; \\
B'^x &= B^x \\
B'^y &= B^y \cosh\zeta + \frac{E^z}{c} \sinh\zeta \\
B'^z &= B^z \cosh\zeta - \frac{E^y}{c} \sinh\zeta.
\end{aligned}$$

These can also be written as

$$\begin{aligned}
\mathbf{E}' &= \gamma(\mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \\
\mathbf{B}' &= \gamma(\mathbf{B} - \frac{1}{c}\boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}).
\end{aligned}$$

8.5 The Dual Field-Strength Tensor

Another useful tensor whose components are the electric and magnetic field-strengths is the *dual* tensor defined by

$${}^*F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$$

which makes use of the *Levi-Civita alternating tensor* $\epsilon_{\alpha\beta\mu\nu}$:

$$\epsilon_{\alpha\beta\mu\nu} = \begin{cases} \pm 1 & \text{if } \alpha\beta\mu\nu \text{ is a } \pm\text{ve permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

Thus ${}^*F_{01} = \frac{1}{2}(\epsilon_{0123}F^{23} + \epsilon_{0132}F^{32}) = F^{23} = -B^x$, etc., and ${}^*F_{12} = \frac{1}{2}(\epsilon_{1203}F^{03} + \epsilon_{1230}F^{30}) = F^{03} = -E^z/c$, etc., so that

$$||{}^*F_{\alpha\beta}|| = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z/c & E^y/c \\ B^y & E^z/c & 0 & -E^x/c \\ B^z & -E^y/c & E^x/c & 0 \end{pmatrix}.$$

To show that ${}^*F$ is in fact a tensor, it is simplest to show that $\epsilon_{\alpha\beta\mu\nu}$ is a tensor, and this is done by consideration of the formula

$$\begin{aligned}
\epsilon_{\alpha\beta\mu\nu} \to {\epsilon'}_{\alpha\beta\mu\nu} &= \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\epsilon_{\gamma\delta\rho\sigma} \\
&= \det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\alpha\beta\mu\nu}.
\end{aligned}$$

The determinant is just $[\det\Lambda]^{-1}$, which from the result obtained earlier is ± 1. Its presence shows that ϵ is in fact a *pseudo*-tensor; it changes sign under a reflection or a change in the sense of time. Lorentz transformations like these, which have $\det\Lambda = -1$ are called *improper*, and we will henceforth exclude them. For the *proper* Lorentz transformations, the determinant factor is unity.

Since F and ${}^*F$ are both tensors, the contractions $F^{\mu\nu}F_{\mu\nu}, F^{\mu\nu}{}^*F_{\mu\nu}$ and ${}^*F^{\mu\nu}{}^*F_{\mu\nu}$ are Lorentz *scalars.* It is easy to evaluate them; they are respectively $2(\mathbf{B}^2 - \mathbf{E}^2/c^2), 4\mathbf{E}\cdot\mathbf{B}/c$, and $2(\mathbf{E}^2/c^2 - \mathbf{B}^2)$. In fact these are the *only* independent Lorentz scalars which can be constructed from the field-strengths. Note that they allow a classification of electromagnetic field configurations according as $E^2 > B^2c^2$, $E^2 = B^2c^2$, $E^2 < B^2c^2$ which is invariant under Lorentz transformations.

8.6 The Energy-Momentum Tensor

Recall the definition of the Maxwell stress tensor, here denoted T^M_{ab}, in an earlier lecture:

$$\frac{1}{\epsilon_0}T^M_{ab} = E_a E_b + c^2 B_a B_b - \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\delta_{ab}.$$

The divergence of this expression was directly related to the conservation of momentum, with $\mathbf{g} = \frac{1}{c^2}(\mathbf{E}\times\mathbf{H})$ the electromagnetic momentum density. Since Maxwell's equations are invariant under the transformations of special relativity, and can be written in the manifestly covariant form $\partial^\mu F_{\mu\nu} = 0$, there must exist a covariant form of the stress tensor, which includes the expression above. Setting $\epsilon_0 = 1$ for simplicity in the following, it is easy to guess that this expression is

$$\frac{1}{c^2}T^{\mu\nu} = -F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}.$$

The components of this tensor are

$$\begin{aligned} T^{00} &= \frac{1}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2), \\ T^{0b} &= -c(\mathbf{E}\times\mathbf{B})_b, \\ -T^{ab} &= E_a E_b + c^2 B_a B_b - \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\delta_{ab}. \end{aligned}$$

Notice that the T^{00} component is equal to the energy density $\mathcal{E}$ and the vector defined by the T^{0a} components is $-c$ times the momentum density $\mathbf{g}$. Finally, T^{ab} is minus the Maxwell tensor above.

It is straightforward to check that the condition

$$\partial^\mu T_{\mu\nu} = 0$$

follows using the equations of motion $\partial^\mu F_{\mu\nu}$. This equation expresses in a Lorentz covariant manner the conservation of energy and momentum. Integrating it over a closed spacelike hypersurface yields the result that the energy flow across the surface is balanced by the change of energy within, and similarily for the momentum.

In the presence of currents due to charged particles, the Maxwell equations become $\partial^\mu F_{\mu\nu} = \mu_0 j_\nu$. The conservation law here becomes

$$\partial^\mu T_{\mu\nu} = -F_{\nu\rho}j^\rho.$$

(This is mostly easily derived using the Lagrangian formulation of electrodynamics, as we will see later in the course.) The components of this conservation law are

$$\begin{aligned} \frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{P} &= -\mathbf{J}\cdot\mathbf{E}, \\ \frac{\partial g_a}{\partial t} - \partial^b T^M_{ab} &= -\rho\mathbf{E}_a - (\mathbf{J}\times\mathbf{B})_a. \end{aligned}$$

The first equation expresses the conservation of energy. The second expresses conservation of momentum. The sources are described by the four-vector $j^\mu = (c\rho, \mathbf{J})$ and $\mathcal{P} = \mathbf{E}\times\mathbf{H}$. The four-vector $f^\beta = F^{\beta\alpha}j_\alpha$ is the *Lorentz force density*, with components

$$f^\beta = (\frac{1}{c}\mathbf{J}\cdot\mathbf{E}, \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}).$$

Note that this discussion has assumed that the charged particles do not radiate or have self-interaction effects due to the electromagnetic field. In general (eg at relativistic velocities or energies), this assumption cannot be made. A full relativistic treatment of the interaction of particles and fields requires *quantum field theory*; for electrons and the electromagnetic field this is the theory of *quantum electrodynamics*, or *QED* for short. Some discussion of this will be given at the end of the course.