Scientific Measurement

PHY-4103

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Lecture 6 - The Gaussian Probability Distribution

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Last lecture we looked at the Binomial probability distribution:

$$f(k;n,p) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= {}_nC_k p^k (1-p)^{n-k}$$

$$= {}_nC_k p^k q^{n-k} \quad \text{where } q = 1-p \quad \text{q=P(fail)}$$

Prob. of k successes given n trials,
and probability of success for single trial = p

Used to determine uncertainties in situations where each trial has
a true/false or yes/no result, e.g. opinion polls & efficiency calcs.

Another common probability distribution: The Poisson Distribution

Relevant for counting experiments

Describes statistics of events occurring at a random but at a definite rate e.g.

number of radioactive decays in 1 min interval

number of births in UK per 24 hours

Experiment: count number of radioactive decays in 1 min interval: k

Assume 100% efficient detector - no error on k

Repeat experiment → get different k ⇒ statistical fluctuation!

n nuclei	$n \sim 10^{20}$
p prob of single decay in 1 min	$p \sim 10^{-20}$
k decays measured in 1 min	$n \times p \sim$ finite

Can use binomial distribution in simplified form ⇒ Poisson Distribution

As $n \to \infty$ and $p \to 0$ with $np \to \lambda$ (const) then binomial dist gives Poisson dist.

Probability of observing k counts if mean number of counts is λ

$$f(\lambda,k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$\lambda = n \times p$ = mean no. of counts

k = observed no. of counts

Important properties of the Poisson distribution:

$$\lambda = \bar{k}$$

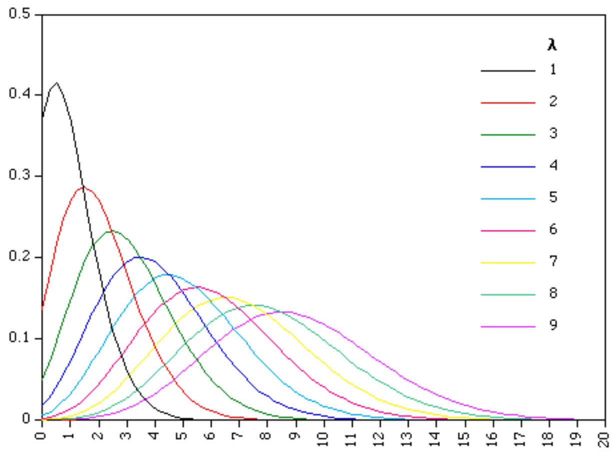
$$\sigma = \sqrt{\lambda}$$

In other words:

measurement of k gives estimate of λ

measurement of $\sqrt{k}$ gives estimate of σ

i.e. the <u>true</u> mean is given by the mean of k and also k gives the true variance σ^2



Poisson dist. skewed for small λ

More symmetric as λ increases

1 measurement gives λ and σ

$$k \pm \sqrt{k}$$

fractional error: $\frac{\sigma}{\lambda} = \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}$

Thus error decreases as k increases but only as the square root!

The most important probability distribution: Gaussian

Other names: Bell curve, Normal Distribution

Binomial & Poisson dists. are discrete
Gaussian is continuous probability distribution

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ = mean of distribution
σ = standard deviation
x = measured variable

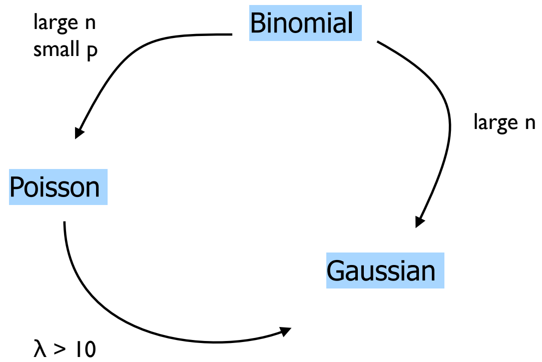
normalisation factor $\int_{-\infty}^{+\infty} G(x)dx = 1$

Gaussian distribution has nice properties:

mean = median = mode: $\mu = \int_{-\infty}^{+\infty} xG(x)dx = 1$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 G(x)dx$$

- Gaussian dist. is symmetric about mean μ
- μ shifts the peak of distribution



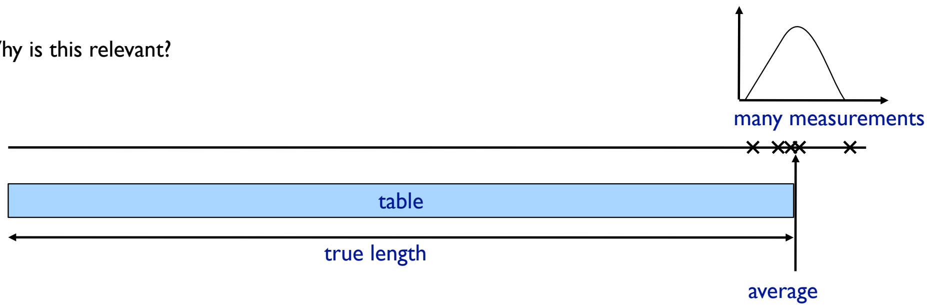
Why is the Gaussian Distribution so important?

The Central Limit Theorem

The mean of a large number of independent samples each of size n from the same probability distribution (not necessarily Gaussian) has approximately a Gaussian distribution centred on the population mean and variance which reduces as 1/n

http://www.intuitor.com/statistics/CLAppClasses/CentLimApplet.htm

Why is this relevant?



Return to the measurement of our table
Repeat this 100 times
We will see a spread of measurements
Spread arises from many small random effects
Central Limit Theorem tells us that the spread will be Gaussian

This explains why Gaussian errors appear everywhere!

Confidence Intervals

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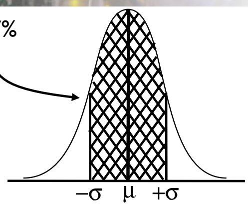
TABLE 3.2
TWO-TAILED GAUSSIAN INTEGRAL
Giving the percentage probability that a point lies within the given number of σ from the mean

$\frac{x-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.10	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.20	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.30	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.40	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.50	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.60	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.70	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.80	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.90	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.00	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.10	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.20	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.30	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.40	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.50	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.60	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.70	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.80	92.81	92.97	93.12	93.28	93.42	93.57	97.71	93.85	93.99	94.12
1.90	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.00	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.10	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.20	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80
2.30	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.40	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72

Area=68.27%



Total area under the curve = 1
Normalisation condition!
Integrating part of the curve tells us of probability
E.g. prob of result within ±σ

68.3% within ±1σ
95.5% within ±2σ
99.7% within ±3σ

Thus measurement quoted as

63 ± 1 cm means

68% prob within 62-64 cm
95% prob within 61-65 cm
99% prob within 60-66 cm

TABLE 3.3
ONE-TAILED GAUSSIAN INTEGRAL
Giving the percentage probability that a point lies within the given number of σ to one side of the mean

$\frac{x-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	50.00	50.40	50.80	51.20	51.60	51.99	52.39	52.79	53.19	53.59
0.10	53.98	54.38	54.78	55.17	55.57	55.96	56.36	56.75	57.14	57.53
0.20	57.93	58.32	58.71	59.10	59.48	59.87	60.26	60.64	61.03	61.41
0.30	61.79	62.17	62.55	62.93	63.31	63.68	64.06	64.43	64.80	65.17
0.40	65.54	65.91	66.28	66.64	67.00	67.36	67.72	68.08	68.44	68.79
0.50	69.15	69.50	69.85	70.19	70.54	70.88	71.23	71.57	71.90	72.24
0.60	72.57	72.91	73.24	73.57	73.89	74.22	74.54	74.86	75.17	75.49
0.70	75.80	76.11	76.42	76.73	77.04	77.34	77.64	77.94	78.23	78.52
0.80	78.81	79.10	79.39	79.67	79.95	80.23	80.51	80.78	81.06	81.33
0.90	81.59	81.86	82.12	82.38	82.64	82.89	83.15	83.40	83.65	83.89
1.00	84.13	84.38	84.61	84.85	85.08	85.31	85.54	85.77	85.99	86.21
1.10	86.43	86.65	86.86	87.08	87.29	87.49	87.70	87.90	88.10	88.30
1.20	88.49	88.69	88.88	89.07	89.25	89.44	89.62	89.80	89.97	90.15
1.30	90.32	90.49	90.66	90.82	90.99	91.15	91.31	91.47	91.62	91.77
1.40	91.92	92.07	92.22	92.36	92.51	92.65	92.79	92.92	93.06	93.19
1.50	93.32	93.45	93.57	93.70	93.82	93.94	94.06	94.18	94.29	94.41
1.60	94.52	94.63	94.74	94.84	94.95	95.05	95.15	95.25	95.35	95.45
1.70	95.54	95.64	95.73	95.82	95.91	95.99	96.08	96.16	96.25	96.33
1.80	96.41	96.49	96.56	96.64	96.71	96.78	96.86	96.93	96.99	97.06
1.90	97.13	97.19	97.26	97.32	97.38	97.44	97.50	97.56	97.61	97.67
2.00	97.72	97.78	97.83	97.88	97.93	97.98	98.03	98.08	98.12	98.17
2.10	98.21	98.26	98.30	98.34	98.38	98.42	98.46	98.50	98.54	98.57
2.20	98.61	98.64	98.68	98.71	98.75	98.78	98.81	98.84	98.87	98.90
2.30	98.93	98.96	98.98	99.01	99.04	99.06	99.09	99.11	99.13	99.16
2.40	99.18	99.20	99.22	99.25	99.27	99.29	99.31	99.32	99.34	99.36

Can integrate up to some max value
One tailed Gaussian
integral = 0.5 + ½ (area $-\sigma$ to $+\sigma$)
84.1% < 1σ
97.7% < 2σ