Laboratory Exercise 2 – LINEAR and NON-LINEAR BEHAVIOUR

Introduction

Most students know something about the behaviour of elastic materials. They know that if a length of wire is pulled it stretches, and that for small forces (or loads, since the force is often applied by hanging a weight) the extension is proportional to the load; when the load is removed the wire reverts to its original length. If the load is too large, the wire becomes permanently stretched, and for sufficiently large loads it breaks.

The statement that extension x is proportional to applied force F, so that a graph of x versus F is a straight line, is Hooke's law. However, this so-called 'law' is nothing like Newton's laws of motion, which hold for any object; it is a description of what Hooke and others found to be the case for a few common substances, particularly metals. But for many materials the 'law' is a poor description of their behaviour under stress. Quite often it *appears* to be obeyed but careful measurements show that a graph of x versus F is not exactly a straight line. Observations of such **non-linear** behaviour can reveal a lot about the molecular structure of the material.

In this exercise you will study the stretching of a rubber band and find a better description of the relation between F and x than the Hooke's law, $F = Kx$. To do this, your measurements of extension have to be much more precise than the nearest millimetre, but to appreciate this fully you will start by measuring x as a function of F using a simple metre rule.

Extension measured with a metre rule

• The rubber band is hung from a thin rod and supports a weight hanger on which weights can be loaded — see figure 1. With a second stand clamp a metre rule so that the position of the bottom of the weight hanger can be read off on the millimetre scale.

• Record the scale readings as you add weights to the weight hanger, in steps of 20 grams up to 200 grams (larger loads have been found to result in permanent stretching of the band). Remember that the weight hanger itself has a mass of 20g. You should be able to estimate the reading to the nearest half-millimetre.

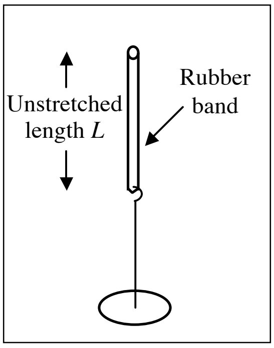

Figure 1 Set-up

• Plot your results directly (see footnote[1]) as mass in grams on the the horizontal axis (**abscissa**) versus scale reading on vertical axis (**ordinate**).

• If you have been careful, your readings of the hanger position are probably accurate to within about a half-millimetre. To represent this draw vertical **error bars** on each side of your measured points to indicate that their uncertainty is ±0.5 mm. The value of the mass is known comparatively accurately (unless you made a mistake in reading the numbers!) so no horizontal error bars are needed.

You will probably find that you can draw a straight line passing close to the points, through the majority of the error bars. You are unlikely to notice any obvious curvature or other systematic deviation from a straight line. In other words, *within the accuracy of your measurements* the behaviour of the rubber band is **linear** — it obeys Hooke's law.

[1]**Footnote:** The mass m is of course directly proportional to the force mg. Your scale reading is not itself the *extension* since it is measured from some arbitrary position x_o on the ruler, and it may decrease rather than increase, depending on which way up the ruler is clamped, but these factors will only change $F = Kx$ to $F = K(x_o - x)$, which is still a linear (straight-line) relation.

Measurement Using a Travelling Microscope

To detect small departures from linear behaviour, you need to measure the extension more accurately. We shall use a **travelling microscope** focused on the weight hanger to follow the stretching of the rubber band, and a **digital** scale to measure the distance moved by the microscope.

• Try using the travelling microscope and learn how to operate it. It is capable of recording to an accuracy of 0.01 mm. Focus the microscope on the hanger so that the horizontal cross-wire is aligned with the bottom of the weight hanger. Check that the microscope can move downwards at least three centimetres, far enough to view the mark even when the band is fully extended. *Be careful not to confuse millimetres and centimetres.*

• Now repeat your measurements of extension versus load, adding mass in increments of 20 grams up to 200 grams. Measure and record the extension as accurately as you can. As you take the measurements, plot them as scale reading versus mass.

• Measure L, the unstretched length of the rubber band by placing the band on a flat surface and put a ruler or pen on top so that it lies flat. Then use the travelling microscope to measure this horizontal length of the band.

What do you find? Probably your data points lie on a gentle curve, not on a straight line at all — if this isn't obvious, hold the graph nearly level with your eye and look along the line of points. With the increased precision of measurement, the elastic behaviour is clearly non-linear.

Further investigation of the non-linear behaviour

If the expression $F = Kx$ is inappropriate, can we find a better description? Suppose the true relation is a power series:

$$F = A\left(\frac{x}{L}\right) + B\left(\frac{x}{L}\right)^2 + C\left(\frac{x}{L}\right)^3 + \ldots$$

Note that we have divided the extension x by the original un-stretched length L to give the *fractional* extension, often called the **strain** — this ensures that every power term of x/L is dimensionless so the coefficients A, B, C, … all have the units of force. You have seen that the linear 'law', taking just the first term of the series, is quite a good first approximation, so let us investigate the effect of adding just the second (**quadratic**) term. Since x is much less than L this term will be much less than the first, even if the coefficients A and B are similar. Your results probably show that F increases slower than a straight line as the extension x increases, which we can achieve by making the coefficient B negative. If we also try putting A numerically equal to B we have the simplest possible non-linear expression:

$$F = A\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right]$$

• Tabulate $(x/L) - (x/L)^2$ and plot this as the ordinate versus the mass m as abscissa. Draw your own conclusions.

Some final algebra

Even more precise investigations have shown that the expression:

$$F = K\left[\left(\frac{S}{L}\right) - \left(\frac{L}{S}\right)^2\right]$$

is a good description of the elastic behaviour of rubber. Here S stands for the stretched length, $L + x$.

• By expanding this expression as a power series in x/L try to **derive** a series in which the simple quadratic expression you used above appears as the first two terms. You will need to use the binomial theorem in your expansion; it is as follows:

$$(1\quad y)^n \quad 1 \quad ny \quad \frac{n\,n\quad 1}{2!}y^2 \quad \frac{n\,n\quad 1\;n\quad 2}{3!}y^3$$

where we must have $y^2 < 1$ and n can be positive or negative.

• How much more accurate would your measurements have to be if you wanted to show experimentally that the next (cubic) term in the expansion was needed? Hint: roughly estimate the size of this term for largest masses you have used.