## BSc/MSci EXAMINATION

## QUANTUM MECHANICS AND SYMMETRY

Time Allowed: 2 hours 30 minutes

Date: $\quad 24^{\text {th }}$ May, 2011

Time: 10:00-12:30

Instructions: Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section $B$ carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises $20 \%$ of the final mark.

A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Numeric calculators are/are not permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. Rodolfo Russo (MO), Dr. Sanjaye Ramgoolam (DMO)

This (reverse cover) page is left blank.

## SECTION A. Attempt answers to all questions.

A 1 Consider the set of all possible permutations of three objects. This set is a group where the operation combining two elements is just the consecutive action of the permutations. Provide an example showing that this group is non-Abelian.

A 2 Consider the space $M_{2}$ of the $2 \times 2$ real matrices as a vector space over the real numbers. Find a set of linearly independent elements.

A 3 Consider the following operators acting on the vector space $\mathbf{C}^{2}$ with the standard scalar product

$$
A=\frac{1}{2}\left(\begin{array}{cc}
3 & -1 \\
-1 & 3
\end{array}\right), \quad B=\frac{1}{2}\left(\begin{array}{ll}
1 & 3 \\
3 & 1
\end{array}\right) .
$$

Explain why these operators can represent physical observables. Show that it is possible to specify at the same time the values of both these observables.

A 4 Consider the raising and lowering operators $\hat{a}$ and $\hat{a}^{\dagger}$ whose commutation relation is $\left[\hat{a}, \hat{a}^{\dagger}\right]=1$. Calculate the commutator $\left[\hat{a}^{2},\left(\hat{a}^{\dagger}\right)^{3}\right]$ by writing all the $\hat{a}$ 's as the rightmost factor.

A 5 The ket vector $|n\rangle$ represents the $n^{\text {th }}$ energy level of an harmonic oscillator and is normalised as follows $\langle n \mid n\rangle=1$. Prove that the operator $P=\sum_{n=0}^{\infty}(-1)^{n}|n\rangle\langle n|$ is a unitary operator commuting with the harmonic oscillator Hamiltonian.

A 6 The three operators $\hat{J}_{x}, \hat{J}_{y}, \hat{J}_{z}$ satisfy the angular momentum commutation relations. Suppose that a quantum mechanical system is described by the Hamiltonian is $\hat{H}=$ $c\left(\hat{J}_{x} \hat{J}_{y}+\hat{J}_{y} \hat{J}_{x}\right)$, where $c$ is a constant. Verify that the system is not invariant under rotation along the $z$-axis.

A 7 Consider a bound state of three spin $\frac{1}{2}$ particles. List the possible values for the total spin quantum number of this bound state and the possible outcomes of a measurement of the spin along the $x$-axis.

A 8 Consider an electron whose spin along the $z$-axis is $\frac{\hbar}{2}$. What are the possible values of the spin along the $x$-axis? What are the probabilities of finding each value in a measurement of the spin along the $x$-axis?

A 9 Consider the Hamiltonian operator $\hat{H}$ and the vector $v$ in $\mathbf{C}^{2}$ (use the convention $\hbar=1$ ):

$$
\hat{H}=\left(\begin{array}{ll}
1 & 2 \\
2 & 1
\end{array}\right), \quad v=\binom{1}{0} .
$$

Decompose $v$ along the basis of the energy eigenstates
A 10 If the quantum mechanical system of question (A 9) is described by $v$ at $t=0$ and evolves as dictated by $\hat{H}$, what is the state $v(t)$ describing the system at time $t$ ?

## SECTION B. Answer two of the four questions in this section.

## B1

Consider a quantum mechanical system described by the Hamiltonian $\hat{H}=\hat{p}^{2} / 2+\hat{x}^{2} / 2$ and use the conventions $\hbar=1$ for this problem.
(a) As usual, we can define the operators $\hat{a}=\frac{1}{\sqrt{2}}(\hat{x}+i \hat{p})$ and $\hat{a}^{\dagger}=\frac{1}{\sqrt{2}}(\hat{x}-i \hat{p})$. Show that the coherent state $|\lambda\rangle \equiv \mathcal{N} \exp \left(\lambda a^{\dagger}\right)|0\rangle$ (where $\lambda \in \mathbf{C},|0\rangle$ is the ground state and $\mathcal{N}$ is the overall normalization) is an eigenstate of $a$ and find the corresponding eigenvalue.
(b) Derive the wavefunction, $\psi_{\lambda}(x) \equiv\langle x \mid \lambda\rangle$, corresponding to the coherent state in part (a) (you are not required to derive the overall normalization of $\psi_{\lambda}(x)$ ).
(c) Calculate the average position $x_{\lambda}$ and momentum $p_{\lambda}$ when the state of a harmonic oscillator is described by $|\lambda\rangle$.
(d) Calculate the uncertainty on the position $\Delta x$ and the one on the momentum $\Delta p$. Verify that $\Delta x \Delta p=1 / 2$ for any value of $\lambda$.

## B2

(a) Consider the following operators $\hat{J}_{i}$ on $\mathbf{C}^{4}$ :

$$
\hat{J}_{x}=\frac{\hbar}{2}\left(\begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array}\right), \quad \hat{J}_{y}=\frac{\hbar}{2}\left(\begin{array}{cccc}
0 & -i & -i & 0 \\
i & 0 & 0 & -i \\
i & 0 & 0 & -i \\
0 & i & i & 0
\end{array}\right), \quad \hat{J}_{z}=\hbar\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

This is a realization of the angular momentum algebra in terms of $4 \times 4$ matrices. Choose one of the angular momentum commutation relations and check explicitly that it is satisfied by the $\hat{J}$ 's above.
(b) Show that this representation is reducible.
(c) List the total spin quantum numbers of the irreducible representations contained in the representation in part (b).
(d) Construct an irreducible representation in terms of of $4 \times 4$ matrices.

## B3

(a) The motion of a particle in the 3 -dimensional space is described by the Hamiltonian

$$
\hat{H}=\frac{1}{2}\left(\hat{p}_{x}-B \hat{y}\right)^{2}+\frac{1}{2}\left(\hat{p}_{y}+B \hat{x}\right)^{2}+\frac{1}{2}\left(\hat{p}_{z}^{2}+\hat{z}^{2}\right)
$$

where $B$ is a constant. Check that the angular momentum operator $\hat{L}_{z}$ commute with $\hat{H}$ (you can set $\hbar=1$ in this exercise).
(b) Calculate the commutation relation between the kinetic term in $\hat{H}$, which is $\left(\hat{p}_{x}^{2}+\hat{p}_{y}^{2}+\hat{p}_{z}^{2}\right) / 2$, and $\exp (i B \hat{x} \hat{y})$.
(c) Starting from the eigenvector equation $\hat{H}|\psi\rangle=E|\psi\rangle$ and defining $|\psi\rangle \equiv \exp (i B \hat{x} \hat{y})|\phi\rangle$, derive an equation for $|\phi\rangle$. Show that it has again the form of an eigenvector equation, $\hat{H}^{\prime}|\phi\rangle=$ $E|\phi\rangle$ and that the new Hamiltonian $\hat{H}^{\prime}$ commutes with $\hat{p}_{y}$.
(d) Show that the $\hat{H}^{\prime}$ describe a free particle along the $y$ direction and a harmonic oscillator potential along the $x$ and $z$ directions.

## B4

(a) Consider a particle of mass $m$ that moves freely on a circle of radius $R$. Find the eigenstates of the Hamiltonian $\hat{H}_{0}=\frac{\hat{p}^{2}}{2 m}$ with the usual periodic boundary conditions.
(b) At $t=0$ the system is described by the ket vector $|\psi\rangle$ corresponding to the wavefunction

$$
\psi(x)=A \cos \left(\frac{2 x}{R}\right)
$$

where $A$ is constant and $0 \leq x \leq 2 \pi R$ parametrizes the circle. Write the wave-function in terms of the momentum eigenstates $|n\rangle$ which satisfy $\hat{p}|n\rangle=\frac{n}{R}|n\rangle$ and $\langle n \mid n\rangle=1$, with $n \in \mathbf{Z}$.
(c) A perturbation corresponds to adding

$$
\hat{H}_{\epsilon}=\sum_{k \neq 0} \epsilon^{|k|}[|0\rangle\langle k|+|k\rangle\langle 0|]
$$

to the the free Hamiltonian. Consider $\epsilon$ to be much smaller than all other energies involved in the problem. Use perturbation theory to find the eigenvalues at order $\epsilon^{2}$.
(d) A second identical particle is put in the same circle. Suppose that particles have zero total spin quantum number and that the Hamiltonian is given by the sum of the free Hamiltonians for each particle and an interacting terms

$$
\hat{H}=\frac{\hat{p}_{1}^{2}}{2 m}+\frac{\hat{p}_{2}^{2}}{2 m}+\frac{\hat{p}_{1} \hat{p}_{2}}{2 m} .
$$

Write the eigenvectors for the ground state and first energy level.

## FORMULA SHEET

## Pauli matrices:

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

