Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-325 QUANTUM MECHANICS AND SYMMETRY

Time Allowed: 2 hours 30 minutes

Date: 7^{th} May, 2010

Time: 10:00 - 12:30

Instructions: Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises 20% of the final mark.

A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. Rodolfo Russo (CO), Dr. Sanjaye Ramgoolam (DCO)

SECTION A. Attempt answers to all questions.

A 1 The set of all polynomials $P(x)$ of degree 1 with real coefficients: $P(x) = a + bx$ form a vector space. Are the following three elements $P_1(x) = 1+2x$, $P_2(x) = x$ and $P_3(x) = 1+x$ linearly dependent? Please justify your answer. [5]

A 2 If A is an observable, prove that the eigenalues of A^2 are real and positive. [5]

A 3 Suppose that the commutator between two Hermitian operators $\hat{a}$ and $\hat{b}$ is $[\hat{a}, \hat{b}] = \lambda$, where λ is a complex number. Show that the real part of λ must vanish. [5]

A 4 Δa and Δb represent the uncertainties on simultaneous measurements of the operators $\hat{a}$ and $\hat{b}$ introduced in the previous question. What is the minimal value of the product $(\Delta a)(\Delta b)$? [5]

A 5 Consider the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ whose commutation relation is $[a, a^\dagger] = 1$. Calculate the commutator $[\hat{a}, (\hat{a}^\dagger)^3]$. [5]

A 6 The ket vector $|n\rangle$ represents the n^{th} energy level of an harmonic oscillator and is normalised as follows $\langle n|n\rangle = 1$. Prove that the operator $P = |1\rangle\langle 1| + |3\rangle\langle 3|$ is a projector. [5]

A 7 The three generators of the $su(2)$ algebra $\hat{J}_1$, $\hat{J}_2$, $\hat{J}_3$ satisfy the following commutation relations: $[\hat{J}_k, \hat{J}_l] = i\hbar \sum_{m=1}^{3} \epsilon_{klm} \hat{J}_m$. Calculate the commutator $[\hat{J}_1^2, \hat{J}_2]$. [5]

A 8 A helium atom has two electrons in the first shell (1s). Explain, without detailed derivation, what the value of the total spin quantum number is. [5]

A 9 Consider a spin $\frac{1}{2}$ particle whose spin along the z-axis is $\frac{\hbar}{2}$. What is the probability of finding the value $\frac{\hbar}{2}$ in a measurement of the spin along the x-axis? [5]

A 10 Consider the following Hermitian operator in C^2:

$$H = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} .$$

Find the eigenvalues of the operator e^{iH}. [5]

SECTION B. Answer two of the four questions in this section.

B1

Consider a quantum mechanical system described by the Hamiltonian $\hat{H} = \hat{p}^2/2 + \hat{x}^2/2$ and use the conventions $\hbar = 1$ for this problem.

(a) Write the above Hamiltonian in terms of the operators $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$ and find the energy of the ground state $|0\rangle$. [5]

(b) At $t = 0$ the system is described by the state $|v, t = 0\rangle = \mathcal{N}(|0\rangle + \hat{a}^\dagger|0\rangle)$. Normalise this state appropriately and find the vector $|v, t\rangle$ describing the system at later time $t > 0$. [7]

(c) Compute the average position $x(t) = \langle v, t|\hat{x}|v, t\rangle$. and momentum $p(t) = \langle v, t|\hat{p}|v, t\rangle$. [8]

(d) Check explicitly that the average quantities computed in (c) satisfy the classical equations

$$\frac{d}{dt}x(t) = \frac{\partial}{\partial p}H \ , \quad \frac{d}{dt}p(t) = -\frac{\partial}{\partial x}H \ .$$

[5]

B2

(a) Check explicitly that the three matrices

$$L_x = i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad L_y = i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \quad L_z = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

satisfy the commutation relations of the angular momentum algebra. [5]

(b) Consider a particle of spin 1 which is governed by the Hamiltonian $H = a(L_x + L_y + L_z)$. Give a matrix representation of H and find the possible values for the energy. [8]

(c) Find the ket $|g\rangle$ representing the ground state of H. [5]

(d) Suppose that an external perturbation modifies the Hamiltonian so that it becomes $H_{\text{new}} = H + W$, where

$$W = \hbar \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

and $\epsilon \ll a$. Use perturbation theory and calculate the new ground state energy at the order ϵ. [7]

B3

(a) The motion of a particle in the 3-dimensional space is described by the Hamiltonian $\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z$, where

$$\hat{H}_x = \frac{1}{2}\left(\hat{p}_x^2 + \hat{x}^2\right) \quad , \quad \hat{H}_y = \frac{1}{2}\left(\hat{p}_y^2 + \hat{y}^2\right) \quad , \quad \hat{H}_z = \frac{1}{2}\left(\hat{p}_z^2 + \hat{z}^2\right) .$$

Check that the standard angular momentum operators $\hat{L}_x$, is a constant of the motion [5]

(b) By knowing that the ground state wavefunction for H_x is proportional to $e^{-\frac{x^2}{2}}$, write the wavefunction $\psi_0(x,y,z)$ representing the ground state for H (you are not required to fix the normalization of the wavefunctions in this problem). [5]

(c) Check that $\psi_0(x,y,z)$ is an eigenvector of $\hat{L}_y$ and find the corresponding eigenvalue. [5]

(d) If the system is described by the wavefunction $\phi(x) = \partial_x \psi_0(x,y,z)$, what are the possible outcomes for a measurement of $\hat{L}_y$ [10]

B4

(a) Consider a free particle of mass m is constrained in a 1-dimensional interval of size L. Find the eigenstate of the Hamiltonian $H = \frac{\hat{p}^2}{2m}$ with the usual boundary conditions $\psi(0) = \psi(L) = 0$. [7]

(b) Parametrising this interval with with $0 < x < L$, find the at $t = 0$ the system is described by the ket vector $|\psi\rangle$ corresponding to the wavefunction

$$\langle x|\psi\rangle \equiv \psi(x) = A\left(\cos\left[\frac{2\pi x}{L}\right] - 1\right),$$

where A is constant. Normalize the vector to 1. [5]

(c) What is the average value of the position when the system is in the state $|\psi\rangle$? [5]

(d) What is the probability to find the system in the ground state ? [8]

FORMULA SHEET

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$