

Symmetries III

QMS Week 8

Reducible & irreducible representations of $SO(3)$

Previously, we have shown how to construct explicitly a $(2j+1)$ dimensional representation of the Lie Algebra of $SO(3)$, given by $(2j+1) \times (2j+1)$ matrices $D(\hat{J}_i/\hbar)$, $i = 'x, y, z' \leftrightarrow i = 1, 2, 3$.

and of corresponding group elements $D(\hat{R}_i(\theta_i))$ corresponding to rotations by angles θ_i about i^{th} axis.

These actually define irreducible representations of $SO(3)$. What this means is that the $(2j+1) \times (2j+1)$ matrices $D(\hat{J}_i/\hbar)$ cannot, by any choice of basis in the Hilbert Space H_j , be brought into block diagonal form :-

$$[D(\hat{J}_i/\hbar)] \neq \left(\begin{array}{ccccc} & & 2k_i+1 & & \\ & 2k_i+1 & & 2k_2+1 & \\ & & 2k_2+1 & & \\ & & & \ddots & \\ & & & & 2k_n+1 \\ & & & & 2k_n+1 \end{array} \right)$$

(2)

where k_l , $l=1..n$, could be integer or half-integer valued.

If we could choose a basis for H_j such that

$D(\hat{J}_i/\hbar)$ did have the block diagonal form given

above, then conversely it would define a reducible representation of Lie Algebra of $SO(3)$.

Then we would recognize each $(2k_i+1) \times (2k_i+1)$

block as an irreducible representation of $SO(3)$

with $j = k_l$, so that $D(\hat{J}_i/\hbar)$ does not represent

a single particle with \hat{J}^2 eigenvalue $j(j+1)\hbar^2$ but

rather a system where several different values of total angular momentum quantum numbers are present

(3)

given by the values of k_l , $l=1 \dots n$, which could be integer or half integer valued.

One example of a physical system in which the above is realized is if we consider a 2-particle system where each particle carries angular momentum quantum numbers j_1 and j_2 respectively,

This leads us naturally, to consider the addition of angular momentum.

Addition of Angular Momentum.

Let's consider a system of 2 particles labelled by 1, 2. We shall just focus on their behavior under action of the rotation group $SO(3)$.
 (In the last topic of the course, we will consider multiparticle states more generally.)

(4)



pictorially, we can think of each single particle by itself as carrying angular momentum vector $\langle \hat{J} \rangle_1$, $\langle \hat{J} \rangle_2$ where $\langle \rangle_1$, $\langle \rangle_2$ refer to expectation values in 1-particle states corresponding to particle 1 and particle 2 respectively.

What we would like to discover is what are the possible values for total angular momentum of the combined, 2-particle system?

2-particle Hilbert Space

First we need to consider what is the Hilbert Space of the combined particles? Particle 1 is a state in H_j_1 - which is $(2j_1+1)$ dimensional, and must transforms as a $(2j_1+1)$ dimensional irreducible representation of $SO(3)$. Similarly, particle 2.

(5)

is a state in H_{j_2} and forms a $(2j_2+1)$ irred. representation of $SO(3)$.

It follows that the Hilbert space (just focussing on angular momentum degrees of freedom) of combined 2 particle system is tensor product

$$H_{\text{TOT}} = H_{j_1} \otimes H_{j_2}$$

$$\begin{aligned}\text{dimension of } H_{\text{TOT}} &= \dim(H_{j_1}) \times \dim(H_{j_2}) \\ &= (2j_1+1) \times (2j_2+1).\end{aligned}$$

Notation: Let $\vec{J}^{(1)}$ represent angular momentum vector operator acting on particle 1 subspace H_{j_1} and $\vec{J}^{(2)}$ me same thing but acting on particle 2 subspace

H_{j_2} . It follows that

$$\left(\vec{\hat{J}}^{(1)}\right)_{H_{\text{TOT}}} = \vec{\hat{J}}^{(1)} \otimes \vec{I}^{(2)} ; \quad \left(\vec{\hat{J}}^{(2)}\right)_{H_{\text{TOT}}} = \vec{I}^{(1)} \otimes \vec{\hat{J}}^{(2)}$$

(6)

where \hat{I} is just the identity operator,

and H_{TOT} subscript denotes representation of

the operator on total Hilbert space $H_j_1 \otimes H_j_2$.

The total angular momentum $\vec{\hat{J}} = (\vec{\hat{J}}^{(1)} \otimes \hat{I}^{(2)}) + (\hat{I}^{(1)} \otimes \vec{\hat{J}}^{(2)})$

Here $\vec{\hat{J}}_i^{(1)}$ and $\vec{\hat{J}}_j^{(2)}$ separately satisfy the

$SO(3)$ Lie Algebra commutation relations:-

$$[\frac{\hat{J}_i^{(1)}}{\hbar}, \frac{\hat{J}_j^{(1)}}{\hbar}] = i \epsilon_{ijk} \frac{\hat{J}_k^{(1)}}{\hbar}; [\frac{\hat{J}_i^{(2)}}{\hbar}, \frac{\hat{J}_j^{(2)}}{\hbar}] = i \epsilon_{ijk} \frac{\hat{J}_k^{(2)}}{\hbar}$$

AND $[\frac{\hat{J}_i^{(1)}}{\hbar}, \frac{\hat{J}_j^{(2)}}{\hbar}] = 0 \quad \forall i, j$. In these relations $\frac{\hat{J}_i^{(1)}}{\hbar}, \frac{\hat{J}_j^{(2)}}{\hbar}$ act on H_{TOT}

This last property simply follows from tensor product

structure of H_{TOT} :

$$\left(\frac{\hat{J}_i^{(1)}}{\hbar}\right)_{H_{TOT}} = \hat{J}_i^{(1)} \otimes \hat{I}^{(2)}; \left(\frac{\hat{J}_i^{(2)}}{\hbar}\right)_{H_{TOT}} = \hat{I}^{(1)} \otimes \hat{J}_i^{(2)}$$

(7)

$$\Rightarrow \left(\begin{array}{cc} \hat{\vec{J}}_i^{(1)} & \hat{\vec{J}}_j^{(2)} \\ \end{array} \right)_{H_{\text{TOT}}} = \left(\begin{array}{cc} \hat{\vec{J}}_i^{(1)} & \hat{\vec{I}}^{(1)} \\ \end{array} \right) \otimes \left(\begin{array}{cc} \hat{\vec{I}}^{(2)} & \hat{\vec{J}}_j^{(2)} \\ \end{array} \right) \\ = \hat{\vec{J}}_i^{(1)} \otimes \hat{\vec{J}}_j^{(2)}$$

and $\left(\begin{array}{cc} \hat{\vec{J}}_j^{(2)} & \hat{\vec{J}}_i^{(1)} \\ \end{array} \right)_{H_{\text{TOT}}} = \left(\begin{array}{cc} \hat{\vec{I}}^{(1)} & \hat{\vec{J}}_i^{(1)} \\ \end{array} \right) \otimes \left(\begin{array}{cc} \hat{\vec{J}}_j^{(2)} & \hat{\vec{I}}^{(2)} \\ \end{array} \right) \\ = \hat{\vec{J}}_i^{(1)} \otimes \hat{\vec{J}}_j^{(2)}$

$$\Rightarrow [\hat{\vec{J}}_i^{(1)}, \hat{\vec{J}}_j^{(2)}] = 0 \quad \text{on } H_{\text{TOT}} \quad \forall i, j.$$

What we wish to know is what are allowed values of total angular momentum $\frac{\hat{\vec{J}}}{\hbar}^2$?

Firstly, it follows from above that

$$\frac{\hat{\vec{J}}_i}{\hbar}$$

satisfy the Lie Algebra of $SO(3)$:-

$$[\frac{\hat{\vec{J}}_i}{\hbar}, \frac{\hat{\vec{J}}_j}{\hbar}] = i \epsilon_{ijk} \frac{\hat{\vec{J}}_k}{\hbar}$$

- proof is left as an exercise - - -

(hint: use $\hat{\vec{J}}_i = (\hat{\vec{J}}_i^{(1)} \otimes \hat{\vec{I}}^{(2)} + \hat{\vec{I}}^{(1)} \otimes \hat{\vec{J}}_i^{(2)})$)

It follows that we can use all of our previously learned derivations and techniques of $\hat{J}^2, \hat{J}_z \dots$ (ladder operators, eigenvalue spectrum etc...)

which were a consequence of satisfying these commutation relations.

Basis for H_{TOT}

As we have learned in the single particle case, it will significantly simplify calculations if we can choose as a basis, eigenvectors of a maximal commuting set of operators, in our case, chosen from

$$\hat{J}^2 \left(= \sum_{i=1}^3 \hat{J}_i^2 \right), \hat{J}^{(1)}, \hat{J}^{(2)}, \hat{J}_3, \hat{J}_3^{(1)}, \hat{J}_3^{(2)}$$

Notice that not all of these commute!

$$\text{e.g. } [\hat{J}, \hat{J}_i] = 0 \quad \forall i$$

$$\text{but } [\hat{J}, \hat{J}_j^{(1)}] \neq 0; \quad [\hat{J}, \hat{J}_i^{(2)}] \neq 0 \quad \begin{cases} \text{Exercise:} \\ \text{check this} \end{cases}$$

$$\text{where } \hat{J} = \hat{J}^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{J}^{(2)} + 2 \hat{J}_i^{(1)} \otimes \hat{J}_i^{(2)}$$

(g)

There are two natural choices for basis vectors of \mathcal{H}_{tot} :

$$1) \quad |jm_j\rangle ; \quad \hat{J}^2 |jm_j\rangle = j(j+1)\hbar^2 |jm_j\rangle$$

$$\hat{J}_z |jm_j\rangle = m_j \hbar |jm_j\rangle$$

$m_j = (-j, \dots, +j)$. [Proof follows usual arguments introducing \hat{J}_{\pm} etc...]

$$2) \quad |j_1, m_{j_1}\rangle \underset{\times}{\hat{J}} |j_2, m_{j_2}\rangle \equiv |j_1, j_2; m_{j_1}, m_{j_2}\rangle$$

where $\hat{J}^{(1)} |j_1, j_2; m_{j_1}, m_{j_2}\rangle = j_1(j_1+1)\hbar^2 |''\rangle$

$$\hat{J}^{(2)} |''\rangle = j_2(j_2+1)\hbar^2 |''\rangle$$

$$\hat{J}_z^{(1)} |''\rangle = m_{j_1} \hbar |''\rangle$$

$$\hat{J}_z^{(2)} |''\rangle = m_{j_2} \hbar |''\rangle$$

where $m_{j_1} = (-j_1, \dots, +j_1)$; $m_{j_2} = (-j_2, \dots, +j_2)$

In order to find what are allowed values of j for given j_1, j_2 we need to find the relation between these two bases.

That is:-

$$\langle j_1, j_2; m_{j_1}, m_{j_2} \rangle = \sum_{j, m_j} \underbrace{\langle j, m_j | j_1, j_2; m_{j_1}, m_{j_2} \rangle}_{\text{'single-particle basis'}} \langle j, m_j | \underbrace{\text{Clebsch-Gordan coefficients}}_{\text{'2-particle basis'}}$$

(here we used completeness relation

$$\sum_{j, m_j} \langle j, m_j | j, m_j \rangle = I$$

We will not determine the Clebsch-Gordan

coeffs in this course (see e.g. Sakuri for detailed derivations) - but will just consider what are the conditions for them to be non-vanishing.

This will be sufficient to determine the allowed values of j (and hence m_j).

First, since $\hat{J}_z = \hat{I}_z^{(1)} \otimes \hat{J}_z^{(2)} + \hat{J}_z^{(1)} \otimes \hat{I}_z^{(2)}$

$$\Rightarrow \langle j, m_j | (\hat{J}_z - \hat{I}_z^{(1)} \otimes \hat{J}_z^{(2)} - \hat{J}_z^{(1)} \otimes \hat{I}_z^{(2)}) | j_1, j_2; m_{j_1}, m_{j_2} \rangle = 0$$

But $\langle j, m_j | \hat{J}_z = \hbar m_j \langle j, m_j |$

(11)

$$\text{and } \hat{\vec{I}}^{(1)} \otimes \hat{\vec{J}}_z^{(2)} |j_1, j_2; m_j, m_{j_2}\rangle = \hbar m_{j_2} |u\rangle$$

$$\hat{\vec{J}_z}^{(1)} \otimes \hat{\vec{I}}^{(2)} |u\rangle = \hbar m_j |u\rangle$$

\therefore Clebsch Gordon coeffs vanish unless

$$m_j = m_{j_1} + m_{j_2}$$

What about relationship between j , j_1 and j_2 ?

Answer is j takes on all values from $j_1 - j_2, \dots, j_1 + j_2$

($(j_1 - j_2) \leq j \leq j_1 + j_2$, and each j differing by 1)

This 'selection' rule can be proven from first principles,
using the algebraic properties of $\hat{\vec{J}}, \hat{\vec{J}}^{(1)}, \hat{\vec{J}}^{(2)}, \dots$

(See Sakuri) - here we can make a consistency check that these are indeed precisely the allowed values for j .

Recall that $\dim(H_{\text{TOT}}) = (2j_1+1) \times (2j_2+1)$

(12)

- which is immediately apparent in
 the basis $|j_1, m_{j_1}\rangle \otimes |j_2, m_{j_2}\rangle$ because $m_{j_1} \in (-j_1, \dots, j_1)$
 and $m_{j_2} \in (-j_2, \dots, +j_2)$.

In the (equivalent) $|j, m_j\rangle$ basis, the number
 of basis vectors is $\sum_j (2j+1)$. because we know
 that $m_j \in (-j, \dots, +j)$ takes on $2j+1$ values.

The sum is over all allowed values of j .

Therefore our consistency check is to show that

$$\sum_{\substack{j=j_1+j_2 \\ j=j_1-j_2}} (2j+1) = (2j_1+1)(2j_2+1).$$

Proof:

LHS is an arithmetic series $\sum_{n=1}^N (a_1 + (n-1)d)$

$$\text{with } a_1 = 2(j_1 - j_2) + 1 \quad ; \quad d = 2, \quad N = 2j_2 + 1$$

(13)

$$\sum_{n=1}^{n=N} a_1 + (n-1)d = \frac{N}{2} (a_1 + a_N)$$

$$\text{where } a_N = a_1 + (N-1)d = 2(J_1 - J_2) + 1 + 2J_2 \times 2$$

$$\therefore \sum_{j=J_1+J_2}^{j=J_1+J_2} (2j+1) = \frac{(2J_2+1)}{2} (2 \times (2J_1 - J_2) + 1) + 4J_2 \\ = (2J_1+1)(2J_2+1) \quad \checkmark$$

Thus we have discovered the 'rules' of addition of angular momentum.

What about the corresponding representations

$D(\hat{J}_i/\hbar)$ of the total angular momentum \hat{J}_i ?

Well answer follows similar arguments we discussed when we asked same question for a single particle with spin j . Then we found that

$$D(\hat{J}_i/\hbar)_{m_j m'_j} = \langle m'_j | \hat{J}_i/\hbar | j, m_j \rangle$$

defined a $(2j+1)$ matrix representation of
 $SO(3)$ Lie Algebra.

14

In the 2 particle case it's the same formulae, but we must remember that now j is not fixed but runs over possible values $(j_1 - j_2, \dots, j_1 + j_2)$.

Let's call $D_j \equiv D_i(\hat{J}_i)$ the irreducible representation corresponding to angular momentum eigenvalue j .

Then we have found :-

N.B. This nice 'block diagonal' form
is only true in the basis given by

$$\{|j m_j\rangle, j \in (j_1 - j_2, \dots, j_1 + j_2)\}.$$

If we work in the (perfectly equivalent)

\downarrow -particle basis : $\{|j_1, m_{j_1}\rangle \otimes |j_2, m_{j_2}\rangle, \quad \}$

$$m_{j_1} \in (-j_1, \dots, +j_1)$$

$$m_{j_2} \in (-j_2, \dots, +j_2)$$

we get an equivalent representation

to the block form, i.e. we get :-

$$D' = O D O^{-1}$$

with O some matrix of dimension

$$[(2j_1+1)(2j_2+1)] \times [(2j_1+1)(2j_2+1)]$$

The entries of this matrix are non other
than the Clebsch-Gordan coefficients

$$\langle j m_j | j_1, j_2; m_{j_1}, m_{j_2} \rangle !$$

$$j = (j_1 - j_2, \dots, j_1 + j_2); m_j = (-j, \dots, +j)$$

Concrete Example:

Consider case $J_1 = J_2 = \frac{1}{2}$. e.g. 2 spin $\frac{1}{2}$ particles. [could be a Meson in particle physics - these are composite particles consisting of quark-antiquarks where each quark have spin $\frac{1}{2}$].

From our formulae, the allowed values of total angular momentum are $j = J_1 - J_2 = \frac{1}{2} - \frac{1}{2} = 0$

up to $J_1 + J_2 = \frac{1}{2} + \frac{1}{2} = 1$, in integer steps.

Hence only two values of j are allowed, $j=0$ and $j=1$.

The representation of $SO(3)$ Lie algebra in this system is therefore reducible and consists of the $j=0$ irreducible and $j=1$ irreducible representations respectively.

Let's consider our two sets of basis vectors.

17

Let's use shorthand notation that

$$|J_1=\frac{1}{2}, J_2=\frac{1}{2}; m_{J_1}, m_{J_2}\rangle \equiv |m_1, m_2\rangle$$

where both M_1 and M_2 take on values $+\frac{1}{2}$, $-\frac{1}{2}$ respectively.

$$\text{Recall that } |m_1, m_2\rangle = |m_1\rangle \otimes |m_2\rangle$$

\uparrow \leftarrow
 $|j_1=\frac{1}{2}, m_1\rangle$ $|j_2=\frac{1}{2}, m_2\rangle$

keeping the notation as simple as possible.

These are 4 basis vectors:

$$|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle, |-\frac{1}{2}, \frac{1}{2}\rangle, |-\frac{1}{2}, -\frac{1}{2}\rangle$$

Recall that we also have our other basis choice
namely $\{|j, m_j\rangle \ j=0, 1\}$

i.e. we also have 4 basis vectors :-

$$|j=0, 0\rangle, |j=1, m_j=1\rangle, |j=1, m_j=0\rangle, |j=1, m_j=-1\rangle$$

(8)

In this latter basis it's obvious how the $j=0, j=1$ irreducible representation appear.

We can also identify these in the first basis

Recall that $m_j = m_{j_1} + m_{j_2}$

is rule we found for relating m_j, m_{j_1}, m_{j_2}

$$m_{j_1} \equiv m_1; m_{j_2} \equiv m_2 \Rightarrow m_j = m_1 + m_2$$

$$|\frac{1}{2}, \frac{1}{2}\rangle; m_j = \frac{1}{2} + \frac{1}{2} = 1 \quad |\frac{1}{2}, \frac{1}{2}\rangle; m_j = -\frac{1}{2} + \frac{1}{2} = 0$$

$$|\frac{1}{2}, -\frac{1}{2}\rangle; m_j = \frac{1}{2} - \frac{1}{2} = 0 \quad |\frac{1}{2}, -\frac{1}{2}\rangle; m_j = -\frac{1}{2} - \frac{1}{2} = -1$$

It follows that $|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle$ must be in basis corresponding to $\boxed{j=1}$.

There are 2 states with $m_j = 0$. Which of these is part of the $\boxed{j=1}$ basis?

To answer this question, we need to consider

(19)

the representation, $D(\hat{J}_i/\hbar)$.

In basis $|m_1\rangle \otimes |m_2\rangle$ of $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$

$$\begin{aligned}\hat{J}_i \text{ acts as: } \hat{J}_i &= \hat{I}^{(1)} \otimes \hat{J}_i^{(2)} \\ &\quad + \hat{J}_i^{(1)} \otimes \hat{I}^{(2)}\end{aligned}$$

and hence

$$\begin{aligned}D(\hat{J}_i/\hbar) &= I_2 \otimes D(\hat{J}_i^{(2)}/\hbar) \\ &\quad + D(\hat{J}_i^{(1)}/\hbar) \otimes I_2 ; \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\text{But } D(\hat{J}_i/\hbar) = \frac{1}{2} \sigma_i$$

because this is standard $[J = \frac{1}{2}]$ representation of

$SO(3)$ Lie Algebra, which we showed in past weeks.

$$\begin{aligned}\therefore D(\hat{J}_i/\hbar) &= I_2 \otimes \frac{1}{2} \sigma_i \\ &\quad + \frac{1}{2} \sigma_i \otimes I_2\end{aligned}$$

which is a 4×4 matrix

(20)

$$\text{Given } \sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{2} \left(I_2 \otimes \sigma_x + \sigma_x \otimes I_2 \right) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \mathcal{D}(\hat{J}_x/\hbar)$$

Similar calculation give:-

$$\mathcal{D}(\hat{J}_y/\hbar) = \frac{1}{2} \begin{pmatrix} 0 & i & i & 0 \\ -i & 0 & 0 & i \\ -i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{pmatrix}$$

$$\mathcal{D}(\hat{J}_z/\hbar) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These form a representation of $SO(3)$

(i.e. $\mathcal{D}(\hat{J}_i/\hbar)$ satisfy Lie Algebra commutation relations - check for yourself.)

It is reducible but, as we hinted at,
it is not in block diagonal form!

Indeed if we calculate $D(\hat{J}^2)$

$$\text{where } \hat{J}^2 = (\hat{J}_{1c}^2 + \hat{J}_y^2 + \hat{J}_z^2)$$

$$(\text{and recalling } D(\hat{J}_i^2) = D(\hat{J}_i) \cdot D(\hat{J}_i))$$

we find

$$D(\hat{J}^2) = \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} + \frac{\hbar^2}{4} \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}$$

$$+ \hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{which is not diagonal.}$$

Since we can take $|1\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$|1\frac{1}{2}\rangle \otimes |1\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |1\frac{1}{2}\rangle \otimes |-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

(22)

$$|-\frac{1}{2}\rangle \otimes |-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

 \Rightarrow

$$D(\hat{J}^2) |-\frac{1}{2}, \frac{1}{2}\rangle = +2\hbar^2 |-\frac{1}{2}, \frac{1}{2}\rangle$$

$$D(\hat{J}^2) |-\frac{1}{2}, -\frac{1}{2}\rangle = +2\hbar^2 |-\frac{1}{2}, -\frac{1}{2}\rangle$$

which tells us what we saw previously that both these belong to the $j=1$ irreducible representation ($j(j+1)\hbar^2 = 2\hbar^2$ for $j=1$).

But $|-\frac{1}{2}, \frac{1}{2}\rangle$ and $|-\frac{1}{2}, -\frac{1}{2}\rangle$ by themselves are not eigenvectors of $D(\hat{J}^2)$.

It is easy to see however, that the orthogonal combinations:-

$$v_1 = \frac{1}{\sqrt{2}} (|-\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, \frac{1}{2}\rangle); v_2 = \frac{1}{\sqrt{2}} (|-\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, \frac{1}{2}\rangle) \in V_2$$

are!

$$D(\hat{J}^2) v_1 = D(\hat{J}^2) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

(23)

$$\text{while } D(\hat{J}^2) V_2 = D(\hat{J}^2) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ +1 \\ 0 \end{pmatrix}$$

$$= 0 !$$

Thus it is the state $\frac{1}{\sqrt{2}} (| \frac{1}{2}, \frac{1}{2} \rangle + | -\frac{1}{2}, \frac{1}{2} \rangle)$

that along with $| \frac{1}{2}, \frac{1}{2} \rangle, | -\frac{1}{2}, -\frac{1}{2} \rangle$ form

the basis for $J=1$; while $\frac{1}{\sqrt{2}} (| \frac{1}{2}, -\frac{1}{2} \rangle - | -\frac{1}{2}, \frac{1}{2} \rangle)$

is $J=0$ state.

Recap :

<u>$J=1$</u>	m_J	<u>$J=0$</u>	m_J
$ \frac{1}{2}, \frac{1}{2} \rangle$	1	$\frac{1}{\sqrt{2}} (\frac{1}{2}, \frac{1}{2} \rangle + -\frac{1}{2}, \frac{1}{2} \rangle)$	0
$\frac{1}{\sqrt{2}} (\frac{1}{2}, -\frac{1}{2} \rangle + -\frac{1}{2}, \frac{1}{2} \rangle)$	0	$\frac{1}{\sqrt{2}} (\frac{1}{2}, -\frac{1}{2} \rangle - -\frac{1}{2}, \frac{1}{2} \rangle)$	0
$ -\frac{1}{2}, -\frac{1}{2} \rangle$	-1		

Finally, the change of basis :-

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \rightarrow \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{j=1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{j=0} \right\}$$

can be achieved by the 4×4 matrix

$$\bar{O}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad O = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

(... the rows of O are just normalized eigenvectors of $D(\hat{J}^2)$, so that $O D(\hat{J}^2) \bar{O}^{-1} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{j=1, j=0}$)

It is now a straightforward exercise to show that in new basis, $D(\hat{J}_i/\hbar)$ are block diagonal :-

$$O D(\hat{J}_x/\hbar) \bar{O}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{but } \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = D_{j=1}(\hat{J}_x/\hbar) \quad (\text{see Symmetries II notes})$$

$$0 D(\hat{J}_y/\hbar)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 & | & 0 \\ -i & 0 & i & | & 0 \\ 0 & -i & 0 & | & 0 \\ \hline 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\text{and } 0 D(\hat{J}_z/\hbar)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ \hline 0 & 0 & 0 & | & 0 \end{pmatrix}$$

That is $D(\hat{J}_i/\hbar) = \begin{pmatrix} D_{j=1}(\hat{J}_i/\hbar) & 0 & & & \\ & \vdots & \vdots & \vdots & \\ 0 & 0 & D_{j=0}(\hat{J}_i/\hbar) & & \end{pmatrix}$

with $D_{j=0}(\hat{J}_i/\hbar) = 0$. (since there are no non-trivial representations of $SU(3)$ of $\dim=1$)

This explicitly verifies the representation
is reducible into $\boxed{j=1}$ and $\boxed{j=0}$
irreducible parts.