

# Symmetries I

QMS Week 5

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In the next weeks we will study different kinds of symmetries and their importance in Quantum Mechanics. Mostly, there will be symmetries that you have already met in your previous QM courses, such as rotational symmetry or parity symmetry. Here, we will revisit these with a view as to their group properties, the concept of 'representations' of a group. We will also consider time-reversal symmetry as another example of a discrete symmetry (like parity) in Q.M.

## 3d Rotations and $SO(3)$ Group

Consider a vector with components  $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$  in  $\mathbb{R}^3$ .

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A rotation by an angle  $\alpha$  about the  
Z-axis can be represented as:-

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the rotated vector  $\vec{v}' = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = R_z(\alpha) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Similarly we can represent a rotation by angle  $\beta$  about y-axis by the  $3 \times 3$  matrix  $R_y(\beta)$  :-

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

and  $R_x(\gamma)$  a  $3 \times 3$  matrix representing rotation by  
angle  $\gamma$  about x-axis :-

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}$$

where  $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  usual Cartesian  
unit vectors.

If it is easy to see that successive rotations about a given axis commute

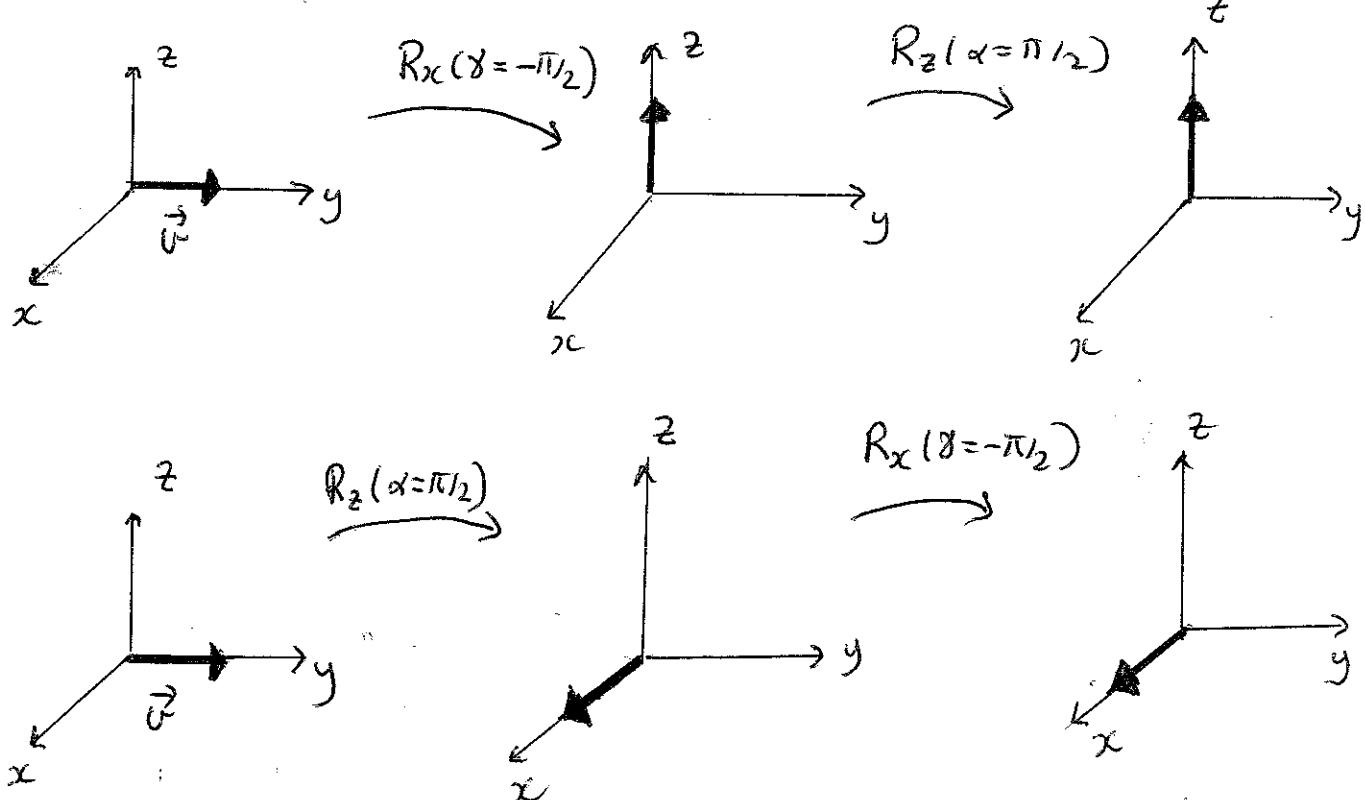
$$\text{eg. } R_z(\alpha) R_z(\alpha') = R_z(\alpha') R_z(\alpha) \\ = R_z(\alpha + \alpha')$$

Similarly for  $R_y$ ,  $R_x$ .

But rotations about different axes do not commute:-

$$\text{eg. } R_z(\alpha) R_x(\gamma) \neq R_x(\gamma) R_z(\alpha).$$

etc..



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Under any rotation, the length of the vector

$\vec{v}$  in  $\mathbb{R}^3$  remains unchanged.

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})} \quad \text{where } (\vec{v}, \vec{v}) \text{ is the inner product}$$

$$\text{on } \mathbb{R}^3, \text{ i.e. } (\vec{v}, \vec{v}) = \vec{v} \cdot \vec{v} = (v_1, v_2, v_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}^T \vec{v}$$

in matrix notation.

Consider  $R$  to be an arbitrary rotation of  $v$

$$v' = Rv \quad \text{and we require that}$$

$$\|v'\| = \|v\| \Rightarrow \sqrt{v'^T v} = \sqrt{v^T v}$$

$$\text{But } v'^T = (Rv)^T = v^T R^T.$$

$$\therefore v^T R^T R v = v^T v$$

$$\Rightarrow \boxed{R^T R = I} \quad I = 3 \times 3 \text{ identity matrix.}$$

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Since the  $3 \times 3$  matrix  $R$  has real entries, the condition  $R^T R = I$  implies

$R$  is an orthogonal matrix  $\underline{R^T = R^{-1}}$

One can easily check  $R_x, R_y, R_z$  certainly satisfy this condition.

How many 'free' parameters can  $R$  have and still be orthogonal? Well a priori,  $R$  has 9 entries (all real)

The condition  $R^T R = I$  fixes 6 of them.

This is because LHS,  $(R^T R)$  is a symmetric matrix  $(R^T R)^T = (R^T R)$ , and a  $3 \times 3$  symmetric matrix has at most 6 independent parameters.

$\therefore$  orthogonal matrices have at most  $9 - 6 = 3$

We can think of these as the 3 arbitrary rotation angles  $\alpha, \beta, \gamma$  introduced above.

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Now it's easy to see that orthogonal matrices form a group under matrix multiplication :-

Let  $R_1, R_2$  be 2 orthogonal matrices:

1) Composition : check  $R_1 R_2$  is also orthogonal:-

$$(R_1 R_2)^T R_1 R_2 = R_2^T \underbrace{R_1^T R_1}_I R_2 = \underbrace{R_2^T R_2}_I = I.$$

So if  $R_1, R_2$  are  $\in$  Group, so is  $R_1 R_2$ .

2) Identity :  $R_1 = I$  -  $3 \times 3$  identity.

2b) Associative  $(R_1 R_2) R_3 = R_1 (R_2 R_3)$  - just follows from fact mat multiplication is associative.

3)  $I R_1 = R_1 I$  for any  $R_1$

4) Inverse : for any  $R_1 \in$  Group,  $R_1^{-1}$  exists

and is also in group . . . Proof :-

$R_1^T R_1 = I$ ; consider  $(R_1^{-1})^T R_1^{-1}$ . Now

$$(R_1^{-1})^T = (R_1^T)^{-1} \quad [\text{proof : } R_1^{-1} R_1 = I \Rightarrow R_1^T (R_1^{-1})^T = I \Rightarrow (R_1^{-1})^T = (R_1^T)^{-1}]$$

$$\text{Hence } (R_i^T)^{-1} R_i^{-1} = (R_i R_i^T)^{-1} = I.$$

So if  $R_i$  is orthogonal,  $R_i^{-1}$  exists and is also orthogonal.

The resulting group is called  $O(3)$

$\xrightarrow{\text{orthogonal}}$        $\xleftarrow{3 \times 3 \text{ matrices}}$

Does this mean that rotations in 3d form the group  $O(3)$ ? Not quite.  $O(3)$  contains not just rotations but also certain reflections

e.g.  $R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  : This is parity transformation  
 $v \rightarrow -v$

Clearly  $R^T R = I$ . But this is not a rotation!

Similarly  $R' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  also is a reflection

$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \rightarrow \begin{pmatrix} -v_1 \\ v_2 \\ v_3 \end{pmatrix}$ ,  $R'^T R' = I$  but  $R'$  is not a rotation

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To restrict the elements of  $O(3)$  to just the rotations, we have to impose an additional constraint, namely  $\boxed{\det R = 1}$

$R_x, R_y, R_z$  all have determinant = 1. However reflections have determinant = -1.

The restriction  $R^T R = I$  AND  $\det R = 1$

defines a subgroup of  $O(3)$  called  $SO(3)$

Special orthogonal

So the 3d rotations form a group  $SO(3)$ .

This group has dimension = 3, in that there are at most 3 independent parameters (the rotation angles  $\alpha, \beta, \gamma$ )

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## Unitary , Uni Modular group U(2)

This is a group which is closely related to SO(3).

Consider complex  $2 \times 2$  matrices of the type

$$U(a,b) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad (\bar{a} \text{ complex conjugate of } a, \bar{b} \text{ " " " } b)$$

Subject to the unimodular condition  $|a|^2 + |b|^2 = 1$

i.e.  $\det U = 1$ .

$$U^+ = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} ; \quad U^+ U = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = I_2$$

Similarly  $U U^+ = 1$ . Hence  $U$  is a unitary matrix

In fact the set  $\{U(a,b) \text{ s.t. } |a|^2 + |b|^2 = 1\}$

forms a group - called SU(2).

Check:

- 1) closure:  $U(a_1, b_1) U(a_2, b_2) = U(a_1 a_2 - b_1 \bar{b}_2, a_1 b_2 + \bar{a}_2 b_1)$

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as can be verified by directly multiplying

$$\begin{pmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{pmatrix} \text{ and } \begin{pmatrix} a_2 & b_2 \\ -\bar{b}_2 & \bar{a}_2 \end{pmatrix}.$$

For closure, we require that  $U(a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$  is also a unitary unimodular matrix.

$$\therefore |a_1 a_2 - b_1 b_2|^2 + |a_1 b_2 + a_2 b_1|^2 = 1$$

$$\begin{aligned} & \quad \swarrow \qquad \searrow \\ |a_1|^2 |a_2|^2 + |b_1|^2 |b_2|^2 & \quad |a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2 \\ - \bar{a}_1 \bar{a}_2 b_1 \bar{b}_2 & \quad + \bar{a}_1 \bar{a}_2 b_1 \bar{b}_2 \\ - \bar{b}_1 b_2 a_1 \bar{a}_2 & \quad + \bar{b}_1 b_2 a_1 \bar{a}_2 \end{aligned}$$

$$\text{LHS} = (\underbrace{|a_1|^2 + |b_1|^2}_{=1})(\underbrace{|a_2|^2 + |b_2|^2}_{=1}) \quad \text{QED} /$$

2) Associativity: follows because matrix multiplication is associative.

3) Identity element exists because  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow a = \bar{a} = 1, b = 0$   
which is unimodular + unitary.

4) Inverse: Given  $U(a, b)$

The inverse element  $\tilde{U}(a, b) = U(\bar{a}, -\bar{b})$

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$U(\bar{a}, -\bar{b})$  is also a unitary matrix and

$$\text{is unimodular since } |\bar{a}|^2 + |-\bar{b}|^2 \\ = |a|^2 + |b|^2 = 1.$$

The group thus formed is called  $SU(2)$

$\begin{matrix} 2 \times 2 \\ \text{matrices} \end{matrix}$

$\begin{matrix} / & \\ \text{special} & \text{unitary} \end{matrix}$

It's called 'special' because  $\boxed{\det U = 1}$

The dimension of the group is 3, since

$a, b$  are 2 complex (4 real) parameters satisfying  
the real constraint  $|a|^2 + |b|^2 = 1 \Rightarrow 3$  independent  
parameters.

$U(2)$  also forms a group - it is just set of  $2 \times 2$   
unitary matrices but without the condition  
 $\det U = 1$ . Any element can be written :-

$$\tilde{U}(a, b, r) = e^{ir} U(a, b) = e^{ir} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$\tilde{U}^+ \tilde{U} = \tilde{U}^+ \tilde{U} = I_2 \quad \text{but } \det \tilde{U} \neq 1.$$

This group has 4 parameters, and  $SU(2)$  is a  
Subgroup.

## Lie Groups / Lie Algebra's

$SO(3)$  and  $SU(2)$  are both examples of something called 'Lie Groups' - or 'continuous' groups.

We have seen they each satisfy the axioms that define a group. But compared to the groups we studied in Week 1,  $SO(3) / SU(2)$  have 3 continuous parameters and arbitrary elements of the group depend on these. In this way Lie Groups are

distinguished from 'discrete' groups - e.g. like the examples in week 1: symmetries of equilateral triangles, or squares etc - these groups had finitely many elements; and there are no continuous parameters involved.

Lie groups have continuously infinitely many elements because they depend on continuous parameters. But we define the dimension of the Lie Group to be the number of independent continuous parameters. E.g.  $SO(3)$  or  $SU(2)$

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This number is 3 as we have seen, so

$SO(3) / SU(2)$  both have dimension 3 as Lie Groups.

## Lie Algebra

Let's go back to the  $SO(3)$  case and recall that the  $3 \times 3$  matrices represent rotations about

$x, y$  or  $z$  axes by angle  $\theta$ :-

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now consider rotation by infinitesimally small

angle  $\theta = \epsilon$ , Taylor Expansions:  $\cos(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^4)$   
 $\sin(\epsilon) = \epsilon + O(\epsilon^3)$

$$R_x(\epsilon) = I_3 + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + O(\epsilon^2)$$

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$$R_y(\epsilon) = I_3 + \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + O(\epsilon^2)$$

$$R_z(\epsilon) = I_3 + \epsilon \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R_x = I_3 - i \in J_x ; R_y = I_3 - i \in J_y$$

$$R_z = I_3 - i \in J_z$$

where  $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} ; J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

$$J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[J_x, J_y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ = i J_z !$$

Notice  $J_x^+ = J_x$  etc.

In fact one can show  $[J_i, J_j] = i \epsilon_{ijk} J_k$

$$J_i = J_x, J_y, J_z \text{ for } i=1,2,3.$$

This is the Lie Algebra of  $SO(3)$ ,

The commutator bracket  $[, ]$  often called Lie bracket.

Thus we have found a connection between

a Lie Group ( $SO(3)$ ) and its Lie Algebra.

It's important to emphasise that Group Structure

is satisfied through matrix multiplication of 2 general rotations  $R_1, R_2$  so that if  $R_1, R_2 \in SO(3)$

so is  $(R_1, R_2)$  as we have shown.

The  $SO(3)$  Lie Algebra is always defined through  
the commutator bracket  $[, ]$ .

Finite group elements by composing many infinitesimal elements

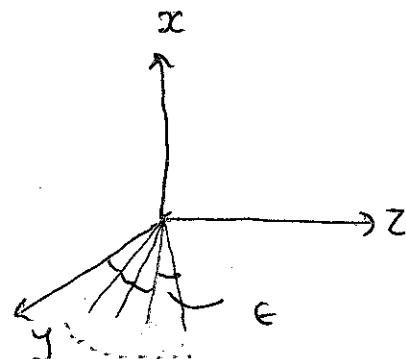
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Consider  $R_x(\epsilon) = (I_3 - i \epsilon J_x)$

- This represents a rotation by tiny angle  $\epsilon$  about  $x$ -axis. Consider now composing many such rotations:-

$$R_x(\epsilon) R_x(\epsilon) \cdots \cdots R_x(\epsilon)$$

$\underbrace{\hspace{10em}}$   
 $N$



In the limit that  $\epsilon \rightarrow 0$  and  $N \rightarrow \infty$

we can, in this way, generate a finite angle rotation about  $x$ -axis:

$$\epsilon = \theta/N$$

$$R_x(\theta) = \lim_{N \rightarrow \infty} \left( I_3 - i \frac{\theta}{N} J_x \right)^N$$

$$= \lim_{N \rightarrow \infty} \sum_{j=0}^N \left( i \frac{\theta}{N} \right)^j (J_x)^j \frac{N!}{j!(N-j)!} \quad (\text{Binomial Expansion})$$

$$\frac{N!}{j!(N-j)!} = \frac{N(N-1)\cdots(N-(j-1)) \times (N-j)!}{j! (N-j)!} \rightarrow \frac{N^j}{j!} \text{ for } N \rightarrow \infty$$

$$\Rightarrow R_x(\theta) = \sum_{j=0}^{\infty} \left( i \frac{\theta}{N} \right)^j J_x^j / j! = e^{-i \theta J_x}$$

We can check this result is correct

explicitly by using the matrix form of  $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$

notice that  $J_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $J_x^3 = J_x$ ,

$$J_x^4 = J_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc...}$$

$$\text{Therefore } \sum_{j=0}^{\infty} (-i\theta)^j \frac{J_x^j}{j!} = \left( \sum_{j=0,2,4}^{\infty} (-i\theta)^j \frac{j}{j!} \right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \sum_{j=1,3,5,\dots}^{\infty} (-i\theta)^j \frac{J_x^j}{j!} = \sum_{j=1,3,\dots}^{\infty} (-i\theta)^j \frac{j}{j!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$\Rightarrow e^{-i\theta J_x} = \cos \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-i \sin \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} //$$

In a similar fashion we can express finite group rotations about  $y, z$  axes in terms of the  $J_y, J_z$  Lie Algebra generators :-

$$R_y(\theta) = e^{-i\theta J_y}$$

$$R_z(\theta) = e^{-i\theta J_z} \quad //$$

Again one may check that using the  $3 \times 3$  matrix representation of  $J_y, J_z$  one can recover the previous expressions for  $R_y(\theta), R_z(\theta)$ .

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Similarly we can link the  $SU(2)$  group

to it's related  $SU(2)$  Lie Algebra.

$$U(a, b) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{write } a = a_1 + i a_2 \quad a_1, a_2, b_1, b_2 \in \mathbb{R}$$

$$b = b_1 + i b_2$$

$$\begin{aligned} U(a, b) &= a_1 I_2 + i b_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i a_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= a_1 I_2 + i \left\{ b_2 \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} + b_1 \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} + a_2 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} \right\} \end{aligned}$$

$$\text{with unimodular constraint } a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$$

$$\text{So if we consider } b_2 = -\epsilon, b_1 = a_2 = 0 \Rightarrow a_1 = 1 - O(\epsilon^2)$$

$$U_x = I_2 - i \epsilon \sigma_x + O(\epsilon^2)$$

$$\text{Similarly taking } b_2 = 0, b_1 = -\epsilon, a_2 = 0 \Rightarrow a_1 = 1 - O(\epsilon^2)$$

$$U_y = I_2 - i \epsilon \sigma_y + O(\epsilon^2)$$

$$\text{and finally choosing } b_1 = b_2 = 0, a_2 = -\epsilon \Rightarrow a_1 = 1 - O(\epsilon^2)$$

$$U_z = I_2 - i \epsilon \sigma_z + O(\epsilon^2)$$

We recognize the Pauli matrices

which satisfy  $[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$

$$\sigma_x, \sigma_y, \sigma_z = \{\sigma_i, i=1,2,3\}.$$

This is a ~~the~~  $SU(2)^{(2)}$  Lie Algebra.  $\underbrace{SO(3)}$

If we define  $J_i = \frac{1}{2}\sigma_i$  then  $\Rightarrow [J_i, J_j] = i \epsilon_{ijk} J_k$

The link with  $SO(3)$  makes us suspect that

$U_x, U_y, U_z$  are related to rotations about  $x, y$  and  $z$

axes. Indeed this is the case - it's just the

realization or 'representation' of the  $SO(3)$  group.

is via  $2 \times 2$  matrices and not  $3 \times 3$  matrices

You have encountered such a representation when

considering spin  $\frac{1}{2}$  systems in Q.M.

As in the  $SO(3)$  case, we can construct finite

group elements  $U_x(\theta), U_y(\theta), U_z(\theta)$  by

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Considering infinite products of infinitesimal group elements:-

$$U_x(\theta) = \lim_{N \rightarrow \infty} \left( I_2 - i \frac{\theta \sigma_x}{N} \right)^N = \exp(-i \theta \sigma_x)$$

Notice  $U^+(\theta) = (e^{-i\theta\sigma_x})^+ = e^{i\theta\sigma_x}$  ( $\sigma_x^+ = \sigma_x$ )

$$\therefore U^+(\theta) U(\theta) = e^{i\theta\sigma_x} e^{-i\theta\sigma_x} = e^{i\theta(\sigma_x - \sigma_x)} = I_2$$

as required.

Also  $U_y(\theta) = e^{-i\theta\sigma_y}$ ;  $U_z(\theta) = e^{-i\theta\sigma_z}$

We refer to the  $\sigma_i$ 's as generators of the Lie Algebra of  $SU(2)$  - and in the previous  $SO(3)$  case, the  $J_i$ 's are generators of the  $SO(3)$  Lie Algebra. From these we can construct finite group transformations by exponentiating as above.

This structure between Lie Groups / Lie Algebra's is not special to  $SU(2)$  /  $SO(3)$  but extends to all Lie Groups / Algebra's.

For example, you may have heard/read mat in the theory of Strong interaction in particle physics (called Quantum Chromodynamics) there is a Lie Group called  $SU(3)$ . This is the group of  $3 \times 3$  unitary matrices with determinant = 1. It has dimension 8

[ a general  $3 \times 3$  complex matrix has 9 complex or 18 real parameters. The condition  $U^\dagger U = I \Rightarrow 9$  real equations  $\Rightarrow 18 - 9 = 9$  parameters left. Finally the condition  $\det U = 1$  removes 1 more degree of freedom  $\Rightarrow 9 - 1 = 8$  free parameters ]

There are 8 generators  $T_i$ ,  $i=1\cdots 8$

and they generate  $SU(3)$  Algebra:  $[T_i, T_j] = i d_{ijk} T_k$   
 $i=1\cdots 8$

The constants  $d_{ijk}$  are known. The quarks and gluons of have very particular transformations under  $SU(3)$  group.