

QMS

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Week 4 : Applications continued.....

Let's now look at some concrete examples.

Free Particle 1d

$$\hat{H} = \frac{\hat{p}^2}{2m} \quad - \text{free massive particle.}$$

Ket $|\psi\rangle$ can be eg. expanded in basis of $|x\rangle$, $x \in \mathbb{R}$.

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \langle x|\psi\rangle |x\rangle$$

where $\langle x|\psi\rangle = \psi(x)$ is time-independent wave function.

Solving eigenvalue equation $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \langle x|\psi_E\rangle = E \langle x|\psi_E\rangle \quad - \text{familiar TISE.}$$

Now since $[\hat{H}, \hat{p}] = 0$ we should also find that

$$|\psi_E\rangle \text{ are eigenstates of } \hat{p}: \quad \hat{p}|\psi_E\rangle = p|\psi_E\rangle$$

$$\Rightarrow -i\hbar \frac{d}{dx} \langle x|\psi_E\rangle = p \langle x|\psi_E\rangle \Rightarrow \langle x|\psi_E\rangle = N e^{-ixp/\hbar}$$

(recall previously)
 $N = \frac{1}{\sqrt{2\pi\hbar}}$

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But as we know $N e^{-i x p / \hbar}$ is not a normalizable wave function! i.e. it is not an element of the Hilbert space of states $L^2(-\infty, \infty)$

which is space of square-integrable complex functions:

$$\text{i.e. } (f, g) \equiv \int_{-\infty}^{\infty} \bar{f}(x) g(x) dx < \infty \equiv \langle f | g \rangle$$

So while $|\Psi_E\rangle = |p\rangle$ solves $\hat{H}|\Psi_E\rangle = E|\Psi_E\rangle$

with $E = p^2/2m$, ($p \in \mathbb{R}$), it is not representing a physical isolated/localized free particle! [since we saw previously $\langle p' | p \rangle = \delta(p' - p) \Rightarrow \langle p | p \rangle \rightarrow \infty!$]

How to remedy this?

Well there are various approaches...

one is to confine the particle to a box so that $x \in [0, a]$ and not $\in (-\infty, \infty)$. This solves the problem about

obtaining normalized states solving $\hat{H}|\Psi_E\rangle = E|\Psi_E\rangle$.

in the following way.

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The Hilbert space is no longer $L^2(-\infty, \infty)$ but

$L^2[0, a]$, for which the momentum eigenstates

$|p\rangle$ have finite norm. Let $\{|x\rangle\}$ be position kets

with $x \in [0, a]$, so that $\int_0^a |x\rangle\langle x| dx = 1$.

$$\Rightarrow \langle p|p\rangle = \int_0^a dx \langle p|x\rangle\langle x|p\rangle = N^2 \int_0^a e^{ixp/\hbar} e^{-ixp/\hbar} dx \\ = N^2 a.$$

Because of the boundaries at $x=0$, $x=a$ we have to impose $\langle x|\Psi_E\rangle \rightarrow 0$ as $x \rightarrow 0$, $x \rightarrow a$.

The solution: $|\Psi_E\rangle = A(|p\rangle - |-p\rangle)$, A some normalization constant.

$$\text{Because } \langle x|\Psi_E\rangle = A (\langle x|p\rangle - \langle x|-p\rangle) \\ = A (e^{ixp/\hbar} - e^{-ixp/\hbar})$$

Obviously $\langle x|\Psi_E\rangle = 0$ at $x=0$, but also

$$\langle x=a|\Psi_E\rangle = 0 \quad \text{if} \quad p = \frac{n\hbar\pi}{a}, \quad n=1, 2, \dots$$

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So we recover the familiar quantization

of momentum p , and $\langle \psi_E | \psi_E \rangle = 1 \Rightarrow A = \sqrt{\frac{\pi \hbar}{a}}$.

Note that now, $|\psi_E\rangle$ is not a momentum

eigenstate as before, but is a linear combination:

$$|\psi_E\rangle = A(|p\rangle - |-p\rangle) \quad \hat{p}|\psi_E\rangle \neq \lambda|\psi_E\rangle \quad * \text{ See } \textcircled{p \neq a} \rightarrow$$

$$\text{but } \hat{H}|\psi_E\rangle = E|\psi_E\rangle \quad ; \quad E = p^2/2m = \frac{\hbar^2 \pi^2}{2ma^2}$$

Another approach is to go back to the particle in 1d

with $x \in \mathbb{R}$ (i.e. $x \in (-\infty, \infty)$) but to consider

not $|p\rangle \rightarrow \int_{-\infty}^{\infty} dp |p\rangle c(p) \equiv |\Delta p\rangle$ and choose

$c(p)$ such that the corresponding wave functions

$$\langle x | \Delta p \rangle \in L^2(-\infty, \infty).$$

* It may seem contradictory that

the states $|\psi\rangle = A(|p\rangle - |-p\rangle)$ are

Eigenfunctions of \hat{H} but not \hat{p} , yet $[\hat{H}, \hat{p}] = 0!$

Actually there is no contradiction!

If a state $|\psi\rangle$ is an eigenstate of 2 operators

$$\hat{A}, \hat{B} : \hat{A}|\psi\rangle = a|\psi\rangle \text{ and } \hat{B}|\psi\rangle = b|\psi\rangle$$

then this obviously implies $[\hat{A}, \hat{B}] = 0$.

But just starting from the condition that $[\hat{A}, \hat{B}] = 0$

does not imply that if say $\hat{A}|\psi\rangle = a|\psi\rangle$ that

this necessarily means $\hat{B}|\psi\rangle = b|\psi\rangle!$ Here is why:-

$$\hat{A}|\psi\rangle = a|\psi\rangle \Rightarrow \hat{B}\hat{A}|\psi\rangle = a\hat{B}|\psi\rangle \\ \hat{A}\hat{B}|\psi\rangle$$

$\Rightarrow (\hat{B}|\psi\rangle)$ is also an eigenstate of \hat{A} with same eigenvalue $a!$

it does not imply $\hat{B}|\psi\rangle = b|\psi\rangle!$

So in our example take $\hat{A} = \hat{H}$; $\hat{B} = \hat{p}$.

$\Rightarrow [\hat{A}, \hat{B}] = 0$. The physical energy eigenstates

was found to be $|\psi_E\rangle = A(|p\rangle - |-p\rangle)$.

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle ; E = p^2/2m$$

All that we can say about $\hat{p}|\psi_E\rangle$ is that

it is a state which also is an eigenstate of \hat{H} with same eigenvalue. Let's check:-

$$\begin{aligned} \hat{p}|\psi_E\rangle &= A(\hat{p}|p\rangle - \hat{p}|-p\rangle) = A(p|p\rangle - (-p)|-p\rangle) \\ &= A(|p\rangle + |-p\rangle) \end{aligned}$$

$$\hat{H}A(|p\rangle + |-p\rangle) = p^2/2m A(|p\rangle + |-p\rangle) ! \Rightarrow \text{DEGENERACY (more than 1 state same energy)}$$

Of course given $[\hat{A}, \hat{B}] = 0$ we can always find

simultaneous eigenvectors - in the above these are just

$|p\rangle$ (or $|-p\rangle$). But by themselves they are not

physical states satisfying the boundary conditions at $x=0, x=a$

only the combination $(|p\rangle - |-p\rangle)$ is.

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This is an example of what we discussed earlier when considering states having continuous eigenvalues $|U_\lambda\rangle$. There we suggested that if $|U_\lambda\rangle \notin$ Hilbert space then $\int_{-\infty}^{\infty} d\lambda c(\lambda) |U_\lambda\rangle$ could be, if $c(\lambda)$ is chosen appropriately. So $|U_\lambda\rangle \Leftrightarrow |p\rangle$ and $\lambda \Leftrightarrow p$.

$$\text{Choose } c(p) = \left[\frac{1}{2\pi(\Delta p)^2} \right]^{\frac{1}{4}} e^{-\frac{p^2}{4(\Delta p)^2}} \quad (\text{i.e. Gaussian in } p)$$

$$\begin{aligned} \Rightarrow \langle x | \Delta p \rangle &= \int_{-\infty}^{\infty} dp \left(\frac{1}{2\pi(\Delta p)^2} \right)^{\frac{1}{4}} e^{-\frac{p^2}{4(\Delta p)^2}} e^{-ipx/\hbar} \\ &= \left(\frac{1}{2\pi(\Delta x)^2} \right)^{\frac{1}{4}} e^{-\frac{x^2}{4(\Delta x)^2}} \end{aligned}$$

$$\text{wim } \Delta p = \hbar/2\Delta x \quad \Rightarrow \quad \Delta p \Delta x = \hbar/2.$$

Thus $|\Delta p\rangle$ represents a state wim minimal uncertainty.

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$$\langle \Delta p | \hat{x} | \Delta p \rangle = \int_{-\infty}^{\infty} dx |\langle \Delta p | x \rangle|^2 x = 0$$

$$\langle \Delta p | \hat{x}^2 | \Delta p \rangle = \int_{-\infty}^{\infty} dx |\langle \Delta p | x \rangle|^2 x^2 = (\Delta x)^2.$$

Similarly $\langle \Delta p | \hat{p} | \Delta p \rangle = 0.$

$$\langle \Delta p | \hat{p}^2 | \Delta p \rangle = (\Delta p)^2$$

with $\langle \Delta p | \Delta p \rangle = 1.$

The states $|\Delta p\rangle$ are NOT energy eigenstates.

If they were, then since $[\hat{H}, \hat{p}] = 0$ they would also be momentum eigenstates. But as we have seen, this leads to problems!

But $|\Delta p\rangle$ does represent a better physical picture of a particle localized in space within a region Δx .

One can also consider how $|\Delta p\rangle$ evolves in time - see p 22/23 of the printed Lecture notes.

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Simple Harmonic Oscillator

classical: $m\ddot{x}(t) = -kx \Rightarrow x(t) = A \cos(\omega t + \phi)$
 $\omega = \sqrt{k/m}$

Quantum: - $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$

and look for solutions $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$.

It is instructive to introduce linear combinations of

$$\hat{x} \text{ and } \hat{p} :- \quad \hat{a} = \sqrt{\frac{m\omega}{\hbar}} \hat{x} + i \sqrt{\frac{1}{2m\hbar\omega}} \hat{p}$$

$$\hat{a}^\dagger = [\hat{a}]^\dagger = \sqrt{\frac{m\omega}{\hbar}} \hat{x} - i \sqrt{\frac{1}{2m\hbar\omega}} \hat{p}$$

(recall $\hat{x}^\dagger = \hat{x}$; $\hat{p}^\dagger = -\hat{p}$) . Coefficients in front of

\hat{x} , \hat{p} chosen so that $[\hat{x}, \hat{p}] = i\hbar \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$

and $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$

To understand the physical interpretation of \hat{a} , \hat{a}^\dagger

consider the state $|\psi_E\rangle$ and act on it with \hat{a} or \hat{a}^\dagger

$$|\psi_E\rangle_+ \equiv \hat{a}^\dagger |\psi_E\rangle \quad ; \quad |\psi_E\rangle_- \equiv \hat{a} |\psi_E\rangle$$

Now consider $\hat{H} |\psi_E\rangle_+ = \hat{H} \hat{a}^\dagger |\psi_E\rangle$

$$= \hat{a}^\dagger \hat{H} |\psi_E\rangle + [\hat{H}, \hat{a}^\dagger] |\psi_E\rangle$$

$$= E |\psi_E\rangle_+ + \hbar\omega \hat{a}^\dagger |\psi_E\rangle$$

$$= (E + \hbar\omega) |\psi_E\rangle_+$$

[using fact that $[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger$]

Similarly $\hat{H} |\psi_E\rangle_- = (E - \hbar\omega) |\psi_E\rangle_-$

Thus \hat{a}^\dagger and \hat{a} are raising / lowering operators for the energy of a state.

Repeating this logic, it is clear that

$$(\hat{a}^\dagger)^m |\psi_E\rangle \text{ is a state whose energy is } (E + m\hbar\omega)$$

$$(\hat{a})^m |\psi_E\rangle \text{ " " " " " } (E - m\hbar\omega).$$

Consider applying \hat{a} repeatedly to $|\psi_E\rangle$. If E is finite and positive eventually we would seem to

discover new states where energy is negative!

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This makes no physical sense so it must

be the case that a state of lowest energy must exist in the system. Call this $|0\rangle$ - the ground state.

Because it has the lowest energy, $\boxed{\hat{a}|0\rangle \equiv 0}$ -

if this were not true we would discover a new state whose energy is lower than $|0\rangle$'s - which contradicts our statement that $|0\rangle$ has lowest energy.

$\hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle$ so ground state has $E = \frac{\hbar\omega}{2}$.

Now by acting repeatedly on $|0\rangle$ with \hat{a}^\dagger - we create new states with higher energies:-

$$|n\rangle \equiv a_n \underbrace{(\hat{a}^\dagger) \dots (\hat{a}^\dagger)}_n |0\rangle \quad ; \quad a_n \in \mathbb{R}.$$

$$\begin{aligned} \hat{H}|n\rangle &= \frac{1}{2}\hbar\omega|n\rangle + \underbrace{[\hat{H}, (\hat{a}^\dagger)^n]}_{n\hbar\omega(\hat{a}^\dagger)^n} |0\rangle \\ &= (n + \frac{1}{2})\hbar\omega|n\rangle \end{aligned}$$

Thus we have found all solutions

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle \quad |\psi_E\rangle = \{|n\rangle\}$$

$$E_n = (n + \frac{1}{2})\hbar\omega.$$

By choosing a_n we can arrange $\langle n|n\rangle = 1$ [$a_n = \frac{1}{\sqrt{n!}}$]

(we define $\langle 0|0\rangle = 1$).

Moreover it is straightforward to show $\langle n'|n\rangle = 0$
 $n \neq n'$

Thus $\{|n\rangle\}$ $n=0, 1, \dots, \infty$ form an orthonormal basis of Hilbert space of states.

Most general state $|\psi\rangle$ in this space is:-

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle ; \quad c_n \in \mathbb{C}.$$

$$\langle \psi' | \psi \rangle = \sum_{n=0}^{\infty} \bar{c}'_n c_n \quad \text{is finite as long}$$

as the \sum is finite. We have encountered this

before in \mathbb{C}^n example of a Hilbert space, but

taking $n \rightarrow \infty$, to get " \mathbb{C}^∞ " = " l^2 " in mathematics literature.

Thus given any state $|\psi\rangle$ (i.e. if c_n are specified) we can compute $\langle\psi|A(\hat{x},\hat{p})|\psi\rangle$ for any observable $A(\hat{x},\hat{p})$, because

we just express $\hat{x} = \hat{x}(\hat{a},\hat{a}^\dagger)$; $\hat{p} = \hat{p}(\hat{a},\hat{a}^\dagger)$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \text{ hence } \langle\psi|\hat{A}(\hat{a},\hat{a}^\dagger)|\psi\rangle$$

can be computed once A is specified.

The time evolution of $|\psi\rangle$ (and $\langle\psi|\hat{A}|\psi\rangle$)

is given as usual through the evolution operator

$$|\psi(t)\rangle = e^{-it\hat{H}/\hbar} |\psi\rangle = \sum_{n=0}^{\infty} c_n e^{-it\omega(n+1/2)} |n\rangle.$$

Wave-functions

Again within the Dirac formalism, we can

make contact with Schroedinger wave function

'picture' or 'basis' through expanding general state

$|\psi\rangle$ in terms of e.g. position kets $|x\rangle$:-

$$|4\rangle = \int_{-\infty}^{\infty} dx \langle x|4\rangle |x\rangle.$$

Let's consider the energy eigenstates $|n\rangle$.

$n=0$; $\langle x|0\rangle = \Psi_0(x)$ = wave function of ground state.

How to determine $\Psi_0(x)$? Well we know

$$\hat{a}|0\rangle = 0 \Rightarrow \langle x|\hat{a}|0\rangle = 0.$$

On the other hand $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}} \hat{p}$

$$\Rightarrow \langle x|\sqrt{\frac{m\omega}{\hbar}} \hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}} \hat{p}|0\rangle = 0.$$

But in the $|x\rangle$ basis $\hat{x}|x\rangle = x|x\rangle$ ($\Rightarrow \langle x|\hat{x} = \langle x|x$)

and $\hat{p} = -i\hbar \frac{d}{dx}$

$$\text{So } x\sqrt{\frac{m\omega}{2\hbar}} \langle x|0\rangle + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \langle x|0\rangle = 0$$

Solution: $\langle x|0\rangle = N e^{-m\omega x^2/2\hbar}$

$$\langle 0|0\rangle = 1 \Rightarrow \int_{-\infty}^{\infty} dx \langle 0|x\rangle \langle x|0\rangle = 1$$

which fixes $N = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

So we have recovered the familiar position-space ground state wave-function:-

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega^2 x^2 / 2\hbar}$$

Notice $\psi_0(x) \in$ Hilbert Space $L^2(-\infty, \infty)$.

To obtain the wave function of ket $|n=1\rangle = |1\rangle$.

$$\psi_1(x) = \langle x | n=1 \rangle. \quad \text{But } |n=1\rangle = \frac{1}{\sqrt{1!}} \hat{a}^+ |0\rangle = \hat{a}^+ |0\rangle.$$

Furthermore
$$\hat{a}^+ = \underbrace{\sqrt{\frac{m\omega}{2\hbar}} \hat{x}}_{\text{position}} - i \underbrace{\frac{1}{\sqrt{2m\hbar\omega}} \hat{p}}_{\text{momentum}}$$

$$\begin{aligned} \Rightarrow \psi_1(x) &= \langle x | \hat{a}^+ |0\rangle = \langle x | \downarrow |0\rangle. \\ &= \sqrt{\frac{m\omega}{2\hbar}} x \langle x | 0 \rangle - \frac{i}{\sqrt{2m\hbar\omega}} \frac{d}{dx} \langle x | 0 \rangle \\ &= \sqrt{\frac{m\omega}{2\hbar}} x \psi_0(x) - \frac{\hbar}{\sqrt{2m\hbar\omega}} \frac{d\psi_0(x)}{dx} \end{aligned}$$

$$\Rightarrow \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(x \sqrt{\frac{m\omega}{2\hbar}} + \frac{m\omega}{\sqrt{2m\hbar\omega}} x\right) e^{-m\omega x^2 / 2\hbar}$$

$$\psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}} \left(\frac{m\omega}{\hbar}\right)^{3/4} x e^{-m\omega x^2 / 2\hbar}$$

To obtain wavefunction for $\langle 0|n\rangle$ $n > 1$,
just repeat procedure iteratively....

Then one recovers the solutions familiar from
solving the Schrodinger equation

Coherent / semi-classical states

We saw for the free particle case that
the Gaussian 'wave packets' corresponding to the
state $|\Delta p\rangle$ were special in that they correspond
to states of minimal uncertainty in that

if we compute $\langle \Delta p | \hat{x} | \Delta p \rangle$, $\langle \Delta p | \hat{x}^2 | \Delta p \rangle$ etc....

we find $\Delta x \Delta p = \hbar/2$ - which is the smallest
it can be. Here these states are closest

approximation we can find in the quantum system
to classical solutions.

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We now wish to derive what these states correspond to in the SHO and this will lead us to consider 'coherent' states.

Let us define these states of minimal uncertainty by ket $|\lambda\rangle$ [N.B. ^{printed} lecture notes use notation $|a\rangle$].

Can we find $|\lambda\rangle$ such that

$$\begin{aligned} \langle \lambda | \hat{x} | \lambda \rangle &= x_{cl}(t) - \text{the classical solution} \\ &= \frac{1}{2} (A e^{i\phi} e^{i\omega t} + A e^{-i\phi} e^{-i\omega t}) \end{aligned} ?$$

We can write $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \hat{a} + \sqrt{\frac{\hbar}{2m\omega}} \hat{a}^\dagger$

$$\text{LHS} = \sqrt{\frac{\hbar}{2m\omega}} (\langle \lambda | \hat{a} | \lambda \rangle + \langle \lambda | \hat{a}^\dagger | \lambda \rangle)$$

It follows that if $\hat{a} | \lambda \rangle = \sqrt{\frac{m\omega}{2\hbar}} A e^{-i\phi} | \lambda \rangle$,

(which also implies $\langle \lambda | \hat{a}^\dagger | \lambda \rangle = \sqrt{\frac{m\omega}{2\hbar}} A e^{+i\phi}$)

we would recover the classical result... (assuming $\langle \lambda | \lambda \rangle = 1$)

So let's suppose $\hat{a} |\lambda\rangle = \lambda |\lambda\rangle$, $\lambda \in \mathbb{C}$.

and try to determine $|\lambda\rangle$.

$|\lambda\rangle \neq |0\rangle$ because $\lambda = 0$. (no good!)

Thus $|\lambda\rangle$ is some general state made from linear combinations of eigenstates $|n\rangle$ $n=0, 1, \dots, \infty$.

But $|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$. So without loss of

generality we can say $|\lambda\rangle \equiv \hat{O}(\lambda, \hat{a}^\dagger) |0\rangle$.

for some operator \hat{O} that depends on λ and \hat{a}^\dagger .

$$\Rightarrow \hat{a} |\lambda\rangle = \lambda |\lambda\rangle \Rightarrow \hat{a} \hat{O} |0\rangle = \lambda |\lambda\rangle = [\hat{a}, \hat{O}] |0\rangle \text{ since } \hat{a} |0\rangle = 0$$

Hence if $[\hat{a}, \hat{O}(\lambda, \hat{a}^\dagger)] = \lambda \hat{O}(\lambda, \hat{a}^\dagger)$ we would be done. Since $[\hat{a}, \hat{a}^\dagger] = 1$. Now we can think of

\hat{a} as $\frac{d}{d\hat{a}^\dagger}$ for purposes of this calculation.

$$\Rightarrow \frac{d}{d\hat{a}^\dagger} \hat{O}(\lambda, \hat{a}^\dagger) = \lambda \hat{O}(\lambda, \hat{a}^\dagger)$$

$$\Rightarrow \hat{O}(\lambda, \hat{a}^\dagger) = \tilde{N} e^{\lambda \hat{a}^\dagger}, \quad \tilde{N} \text{ some constant}$$

$$\Rightarrow |\lambda\rangle = \tilde{N} e^{\lambda \hat{a}^\dagger} |0\rangle$$

$$= \tilde{N} \sum_{n=0}^{\infty} \frac{(\lambda \hat{a}^\dagger)^n}{n!} |0\rangle$$

We could fix \tilde{N} by requiring $\langle \lambda | \lambda \rangle = 1$
 but we won't worry about this right now.

The time-evolved state $|\lambda, t\rangle \equiv e^{-it\hat{H}/\hbar} |\lambda\rangle$

$$= \tilde{N} \sum_{n=0}^{\infty} e^{-it\hat{H}/\hbar} \frac{(\lambda \hat{a}^\dagger)^n}{n!} |0\rangle$$

$$= \tilde{N} \sum_{n=0}^{\infty} e^{-i\omega t(n+\frac{1}{2})} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

(Since $|n\rangle \equiv \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$)

$$\Rightarrow |\lambda, t\rangle = \tilde{N} \sum_{n=0}^{\infty} e^{-i\omega t(n+\frac{1}{2})} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

$$\langle \lambda, t | \hat{x} | \lambda, t \rangle = |\tilde{N}|^2 \sqrt{\frac{\hbar}{2m\omega}} \sum_{m,n=0}^{\infty} \frac{\bar{\lambda}^m \lambda^n}{\sqrt{m!n!}} \left\{ e^{i\omega t(m-n)} \langle m | \hat{a} | n \rangle \right.$$

$$\left. + e^{i\omega t(m-n)} \langle m | \hat{a}^\dagger | n \rangle \right\}$$

Now use properties of our eigenstates $|m\rangle, |n\rangle$:-

$$\langle m | n \rangle = \delta_{mn} ; \hat{a} | n \rangle = \sqrt{n} | n-1 \rangle ; \hat{a}^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle$$

$$\Rightarrow \langle m | \hat{a} | n \rangle = \sqrt{n} \delta_{m, n-1} ; \langle m | \hat{a}^\dagger | n \rangle = \delta_{m, n+1} \sqrt{n+1}$$

$$\begin{aligned} \Rightarrow \langle \lambda | \hat{x} | \lambda \rangle &= |\tilde{N}|^2 \sqrt{\frac{\hbar}{2m\omega}} \left(\lambda e^{\bar{\lambda}\lambda} e^{i\omega t} + \bar{\lambda} e^{\bar{\lambda}\lambda} e^{-i\omega t} \right) \\ &= |\tilde{N}|^2 e^{\bar{\lambda}\lambda} \sqrt{\frac{\hbar}{2m\omega}} \left(\lambda e^{i\omega t} + \bar{\lambda} e^{-i\omega t} \right) \end{aligned}$$

$$\sim A \left(e^{+i\phi} e^{i\omega t} + e^{-i\phi} e^{-i\omega t} \right) \text{ by choosing}$$

λ appropriately (and fixing \tilde{N} by $\langle \lambda | \lambda \rangle = 1$)

Notice that $|\lambda\rangle$ is given by an infinite sum overall all eigenstates $|n\rangle$ - so it represent a 'coherent' motion ...

Exercise

Although we didn't discuss the time-dependent gaussian wave packets $|\Delta p, t\rangle$ when we were looking

at the free particle, it is discussed in the printed notes (p22-23).

Check that $\langle \Delta p, t | \hat{x}^2 | \Delta p, t \rangle = x_{cl}^2(t)$

where $x_{cl}(t) = vt$ (we take starting position $x_0 = 0$)

$$v = \left\{ \langle \Delta p, t | \hat{p}_{2m}^2 | \Delta p, t \rangle \right\}^{1/2}.$$

This is another example of why states of minimal uncertainty are regarded as quasi-classical states.