

Week 2: Linear Operators on Vector Spaces

(1)

Consider 2 vector spaces V, W . A linear operator
is a function, (or map) from $V \rightarrow W$ satisfying:-

$$\forall |x_i\rangle \in V, \underset{i=1,2}{A} (a|x_1\rangle + b|x_2\rangle) = aA(|x_1\rangle) + bA(|x_2\rangle) \in W$$

If $W = V$, map A takes V onto itself.

An example of $V \neq W$
Let $V = P_n$ - space
of degree n polynomials

Then $\forall |x\rangle \in V, A B(|x\rangle) \equiv A(B(|x\rangle))$

$$\underset{\text{composition}}{\sim} A B(|x\rangle) \equiv A(B(|x\rangle))$$

$$A = \frac{d}{dx}, \quad W = P_{n-1}$$

composition
of 2 operators.

For a finite dimensional vector space V , ($\dim V = n$)

there exists a finite set of basis vectors $\{|v_1\rangle, \dots, |v_n\rangle\}$

along which any $|w\rangle \in V$ can be decomposed:-

$$|w\rangle = \sum_{i=1}^n c^i |v_i\rangle$$

For V real (complex), $c^i \in \mathbb{R}^n (\mathbb{C}^n)$. There is in fact
an isomorphism between V and $\mathbb{R}^n (\mathbb{C}^n)$ in this case.

To any $|w\rangle$ we can associate $\begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^n \end{pmatrix}$

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and vice-versa, every element $\begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix}$ corresponds to a $|v\rangle \in V$.

Examples:

For any finite dimensional vector spaces V, W, A can be represented as a matrix. Let $\dim V = n$; $\dim W = m$.

$$|v\rangle \in V; |v\rangle = \sum_{i=1}^n c^i |v_i\rangle. \quad \{|v_1\rangle, \dots, |v_n\rangle\} \text{ basis for } V$$

$$A|v\rangle = \sum_{i=1}^n c^i A|v_i\rangle \quad (\text{linearity of } A)$$

Let W have basis $\{|w_1\rangle, \dots, |w_m\rangle\}$. Then

$$A|v_i\rangle = \sum_{j=1}^m |w_j\rangle a^j_i \Rightarrow A|v\rangle = \sum_{i=1}^n \sum_{j=1}^m a^j_i c^i |w_j\rangle$$

So kets $|v\rangle \in V$ represented by column vectors $\begin{pmatrix} c^1 \\ \vdots \\ c^n \end{pmatrix}$
 kets $|w\rangle \in W$ " " " " " $\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$

The linear operator A represented as $^{m \times n}$ matrix a^j_i :

$$\hat{A} \leftrightarrow a^j_i; \quad a^j_i \stackrel{\text{row}}{\underset{\text{column}}{\equiv}} \langle w_j | A | v_i \rangle$$

(assuming orthonormal basis $\langle w_i | w_j \rangle = \delta_{ij}$).

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This can be extended to case when $n, m \rightarrow \infty$ -
but have to take care! - infinite dimensional matrices

do not necessarily have same properties as finite dimensional
ones.

(See P3a for
another example)

Projection Operators

This is an important type of linear operator

from $V \rightarrow V$ ($P: V \rightarrow V$) with special
property that $P^2 = P$.

E.g. Consider a vector $|v\rangle \in V$ with unit norm:

$$\langle v|v\rangle = 1.$$

$$P_v \equiv |v\rangle\langle v| ; \quad P_v|w\rangle = \underbrace{\langle v|w\rangle}_{\text{number}} |v\rangle, \quad \forall |w\rangle \in V. \\ (\text{n.b. } \neq \langle w|v\rangle !)$$

- Projects an arbitrary vector $|w\rangle \in V$ along 'direction' $|v\rangle$.

$$P_v^2 = (|v\rangle\langle v|)(|v\rangle\langle v|) = |v\rangle\langle v| |v\rangle\langle v| = |v\rangle\langle v| \stackrel{=1}{\sim} P_v.$$

This readily generalizes. If $\{|v_1\rangle, \dots, |v_m\rangle\}$ form
an ortho-normal basis for any subspace $C \subset V$.

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Another Example:

Consider vector space P_n - of polynomials in x of degree n , and operator $A = \left(x \frac{d}{dx} + a \right)$

P_n is a vector space of dimension $= n+1$

$$\begin{aligned} A \text{ is linear: } & \left(x \frac{d+a}{dx} \right) (\alpha_1 P_1(x) + \alpha_2 P_2(x)) \\ &= \alpha_1 \left(x \frac{dP_1}{dx} + aP_1 \right) + \alpha_2 \left(x \frac{dP_2}{dx} + aP_2 \right) \end{aligned}$$

for any two polynomials $P_1, P_2 \in P_n$.

In this case $A : P_n \rightarrow P_n$.

because A maps any $P_i \in P_n$ onto another element of P_n : $P_i = \sum_{i=0}^n a_i x^i$

$$AP_i = \left(x \frac{d+a}{dx} P_i \right) = \sum_{i=1}^n (i a_i + a a_i) x^i \in P_n.$$

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$$P_m = \sum_{i=1}^m |v_i\rangle\langle v_i| \quad \text{Projects any } |w\rangle \in V$$

onto the subspace spanned by $\{|v_i\rangle\}_{i=1 \dots m}$.

A simple consequence is that if we take $m = \dim V$ so that the 'subspace' $\{|v_i\rangle\}$ is in fact the whole of V and $\{|v_i\rangle\}$ defines a basis;

$$\sum_{i=1}^{\dim V} |v_i\rangle\langle v_i| = \mathbb{I} \quad \text{identity operator.}$$

- Often called 'completeness condition'.

[Proof]

$$P_{\dim V} |w\rangle = \sum_{i=1}^{\dim V} c^i P_{\dim V} |v_i\rangle = \sum_{i,j=1}^{\dim V} c^i \underbrace{\langle v_j | v_i \rangle}_{\delta_{ij}} |v_j\rangle = |w\rangle$$

] $\rightarrow P4^a$

Action of linear Operators on bra's

Using the scalar product on V we can

realize that any linear operator acting on V has

a natural induced action of the dual space V^* of linear functionals (or bra's)

(4a)

Representations of Projection Operators

We have seen that $P_m = \sum_{i=1}^m |v_i\rangle\langle v_i|$ with

$\langle v_i | v_j \rangle = \delta_{ij}$, is a projection operator onto subspace spanned by $\{|v_1\rangle, \dots, |v_m\rangle\}$ of the n -dimensional vector space V . As we have seen, there are many ways that vectors $|v\rangle \in V$ are realized or represented.

e.g. $|v\rangle \leftrightarrow n\text{-tuple } \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, c_i \in \mathbb{Q}$ or $|v\rangle \in P_n -$

space of n^m order polynomials etc...

Consider then an example: $V = \text{Space } \mathbb{Q}^2$.

A basis (orthonormal) is $|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\langle v_1 | = (1 \ 0); \langle v_2 | = (0 \ 1).$$

Consider e.g. the projection operator $P_{v_1} = |v_1\rangle\langle v_1|$

- what is the matrix corresponding to it?

Well, it is tensor product of $(1 \ 0)$ with $(1 \ 0)$

$$\text{i.e. } P_1 = (1 \ 0) \otimes (1 \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

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or, in components,

$$(P_1)^i_j = (U_1^i)(U_1)_j \quad (U_1)_j=1 \quad ; \quad (U_1)_j=0$$

$$(U_1^T)^{i=1} = 1 \quad ; \quad (U_1^T)^{i=2} = 0$$

$$\Rightarrow P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{Clearly } P_1^2 = P_1.$$

Similarly we can construct a different projector

operator $P_2 = |U_2\rangle\langle U_2|$ and we find $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

We can also verify the previous completeness relation:

$$\sum_{i=1}^2 |U_i\rangle\langle U_i| = \mathbb{I} = P_1 + P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

All this generalizes: $\mathbb{C}^2 \rightarrow \mathbb{C}^n$, in the obvious way.

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$$\text{Consider } \forall |4\rangle \in V, \quad \langle \phi' |4\rangle \equiv \langle \phi |(A|4\rangle) \\ = \langle \phi |A|4\rangle$$

$\langle \phi | \rightarrow \langle \phi' | \equiv \langle \phi | A$ is linear :-

$$(a \langle \phi_1 | + b \langle \phi_2 |)A = a \langle \phi_1 | A + b \langle \phi_2 | A.$$

Notice here that acting on bra $\langle \phi |$, A acts from right $\stackrel{\leftarrow}{A}$
whereas acting on ket $|4\rangle$ it is from left $\stackrel{\rightarrow}{A}$

Example : finite dimensional Hilbert Space

- Have seen that in this case $|4\rangle \in V \leftrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ - complex

column vector, with A represented by $n \times n$ matrix acting on

$$\text{i.e. } A|V\rangle \rightarrow \sum_{i=1}^n A^j_i c^i \quad (1)$$

Then the corresponding bra vectors, $\langle k|4|$ are row vectors:

$(\bar{c}_1 \dots \bar{c}_n)$ and corresponding action of A is:-

$$\langle V | A \rightarrow \sum_{j=1}^n \bar{c}_j A^j_i \quad (2)$$

N.B. It is same matrix A^j_i in both cases

but in (1) it is 'normal' left matrix multiplication -

where in (2) A^j_i acts as ^{matrix} multiplication from right.

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Representation of general linear operators in Dirac formulation

We have seen how projection operators can be written as combinations of bra/kets :-

$$P_m = \sum_{i=1}^m |v_i\rangle\langle v_i|.$$

A natural question is can we represent any linear operator A acting on $|v\rangle \in V$ as some other combination of $|v_i\rangle\langle v_i|$?

Answer is yes!

We have seen that $|v\rangle = \sum_{i=1}^n c^i |v_i\rangle$

$$\text{so } |v'\rangle = A|v\rangle \Rightarrow |v'\rangle = \sum_{i=1}^n c'^i |v_i\rangle$$

$$\text{where } c'^i = \sum_j A_{ji} c^j \quad \left[\begin{pmatrix} c' \\ \vdots \\ c'_n \end{pmatrix} = \begin{pmatrix} A \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} c \\ \vdots \\ c_n \end{pmatrix} \right] \quad \uparrow \text{matrix } A$$

Now consider operator $|v_j\rangle\langle v^i|$

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$$|v_j\rangle \leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ j \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ position}$$

$$\langle v_i | \leftrightarrow (0 \cdots \underset{i^{\text{th}} \text{ position}}{1} \cdots 0)$$

$$\Rightarrow |v_j\rangle \langle v_i| \leftrightarrow \underbrace{v_j \otimes v_i^*}_{\text{matrix}} \quad (v^i = (v_i)^+)$$

$$\begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}_{j^{\text{th}}}^{i^{\text{th}}}$$

matrix with 0's everywhere except from j^{th} row, i^{th} column which has 1.

Hence the operator :-

$$\hat{A} = \sum_{i,j=1}^n a_{ij} |v_j\rangle \langle v_i| = \begin{pmatrix} a'_1 & \cdots & a'_n \\ \vdots & \ddots & \vdots \\ a''_1 & \cdots & a''_n \end{pmatrix} = \text{matrix } A !$$

But this result goes beyond any particular realization of the bra and ket ... it's independent of any particular representation.

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Thus for any $|v\rangle \in V$; $|v\rangle = \sum_{i=1}^n c^i |v_i\rangle$

$$\begin{aligned}
 \hat{A}|v\rangle &= \sum_{i,j} a^j_i |v_j\rangle \langle v^i|v\rangle \\
 &= \sum_{i,j=1}^n a^j_i \sum_{k=1}^n c^k \underbrace{\langle v^i|v_k\rangle}_{\delta^i_k} |v_j\rangle \\
 &= \sum_{i,j} (a^j_i c^i) |v_j\rangle = \sum_j c^j |v_j\rangle \\
 &= |v'\rangle
 \end{aligned}$$

Notice that the bra corresponding to $|v'\rangle$ is

$$\langle v'| = (\hat{A}|v\rangle)^+ = \langle v|\hat{A}^+$$

and one can verify that $\hat{A}^+ = \left(\sum_{i,j=1}^n a^j_i |v_j\rangle \langle v^i| \right)^+$

$$= \sum_{i,j=1}^n a^{*j}_i |v_i\rangle \langle v^i|$$

and that this operator is just the adjoint of

the matrix $A = \begin{pmatrix} a^1_1 & \dots & a^1_n \\ \vdots & & \vdots \\ a^n_1 & \dots & a^n_n \end{pmatrix}$.

Hermitian , Self-Adjoint Operators

A Hermitian operator acting on V is one which

satisfies:- $(Av_1, v_2) = (v_1, Av_2) \quad \forall v_1, v_2 \in V$

or in Dirac notation: $\langle Av_1 | v_2 \rangle = \langle v_1 | Av_2 \rangle$.

Sometimes A is a hermitian only on a subspace $W \subseteq V$ rather than whole space.

Given that any A has a natural action on bra: $\langle \psi | A$

- we can ask what is the ket corresponding to this transformed bra? Answer is $A^+ |\psi\rangle$ where A^+ is called Adjoint of operator A . [equivalently we could start with ket $A|\psi\rangle$ and then $\langle \psi | A^+$ is the corresponding bra]

$$"(A|\psi\rangle)^+$$

Properties:

$$(A+B)^+ = A^+ + B^+$$

$$\forall a \in \mathbb{C}, (aA)^+ = \bar{a} A^+ \quad (\langle a|x\rangle \leftrightarrow \langle x|\bar{a}\rangle)$$

$$(AB)^+ = B^+ A^+ \quad ((AB)|x\rangle = A(B|x\rangle))$$

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Note that A and A^+ act on different objects.

e.g. if A acts on $|4\rangle \in V$; A^+ is the action on corresponding linear functional $\langle 4|A^+$.

A Hermitian operator is one which is also Self-Adjoint
 [but see below]*

Examples

Finite dimensional case: V is isomorphic to \mathbb{C}^n

Then A can be represented as $n \times n$ matrix acting on n -tuple in \mathbb{C}^n . Then A^+ is represented by simply the Hermitian conjugate of the matrix representing A :

$$A \leftrightarrow \{\alpha_i^j\}; A^+ \leftrightarrow \{(\alpha_i^j)^*\} = \{\alpha_i^{*j}\}$$

In this case, A acts on whole of \mathbb{C}^n and there are

no subtleties.

Consider infinite dimensional example: $|4\rangle \in L^2(-\infty, \infty)$

- square integrable functions, $x \in \mathbb{R}$. Consider position operator $A = x$. It's not guaranteed that $x|4\rangle \in L^2$

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e.g. take $|4\rangle \rightarrow \psi(x) \sim \frac{xc}{1+x^2}$.

Then $xc\psi(x)$ not square integrable on $(-\infty, \infty)$!

Thus one has to restrict L^2 to a smaller subspace

$(L^2)'$ if $A=xc$ is to act so that $A|4\rangle \in (L^2)'$.

Eigen Vectors , Eigenvalues

Let A be a linear operator $A : V \rightarrow V$.

If $|v\rangle \in V$ such that $A|v\rangle = \lambda|v\rangle$

for $\lambda \in \mathbb{C}$, $|v\rangle$ is an eigenvector of A with eigenvalue λ .

It is possible that 2 linearly independent vectors $|v\rangle, |w\rangle$ satisfy $A|v\rangle = \lambda|v\rangle$; $A|w\rangle = \lambda|w\rangle$ - i.e. have same eigenvalue. In general set of all such linearly independent vectors span a subspace of V - called Eigen space.

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Recall properties of finite dimensional case:

If $\dim V = n$, V is isomorphic to \mathbb{C}^n .

$A \leftrightarrow$ matrix a_{ij}^{ij} acting on \mathbb{C}^n .

Evalues of a_{ij}^{ij} obtained from characteristic equation:

$$\det(a_{ij}^{ij} - \lambda \delta_{ij}) = 0 \quad i, j = 1 \dots n.$$

The corresponding eigenvectors are obtained by solving

Set of n linear equations:-

$$\sum_{i=1}^n (a_{ij}^{ij} c^i - \lambda c^j) = 0 \quad j = 1, 2, \dots, n.$$

Important result: Eigenvectors with different eigenvalues

always form a set of linearly independent vectors.

[if $|v\rangle$ and $|w\rangle$ are the two eigenvectors with e-values λ, λ' , $\lambda \neq \lambda'$; If they were not linearly independent then $|v\rangle = a|w\rangle$ for some $a \in \mathbb{C}$. But this implies that $\lambda = \lambda'!$ QED]

Eigenvectors of Self-adjoint Operators

- Hermitian (\Leftrightarrow self-adjoint) matrices have special property in that their eigenvectors form a basis for \mathbb{C}^n , and corresponding eigenvalues are real. Indeed it's also possible to make these eigenvectors into an orthonormal basis:

$$A|\psi_i\rangle = \lambda_i |\psi_i\rangle ; \quad \langle \psi_i | \psi_j \rangle = \delta_{ij} \quad i=1\dots n.$$

$\lambda_i \in \mathbb{R}$. The fact that the vectors $\{|\psi_i\rangle\}$ form a

complete set is same as $\sum_{i=1}^n |\psi_i\rangle \langle \psi_i| = \mathbb{I}$ [Identity operator on V]

[Sketch of proof given in printed notes p16]

$$\begin{aligned} \text{Reality of } \lambda_i \text{ is easy to see: } \langle \psi_i | A | \psi_i \rangle &= \lambda_i \langle \psi_i | \psi_i \rangle \\ &= \overline{\langle \psi_i | A^\dagger | \psi_i \rangle} = \bar{\lambda}_i \langle \psi_i | \psi_i \rangle \end{aligned}$$

$$\Rightarrow \bar{\lambda}_i = \lambda_i.$$

What about the case where V is infinite dimensional?

- as in many examples in Q.M. ?

Well then there are subtleties (as usual!)

- the previous theorem no longer holds - that even if we find solutions to $A|v_i\rangle = \lambda_i|v_i\rangle$ $i=1, 2, \dots, \infty$, they may not necessarily form a complete basis of V .

It turns out there are two situations depending on 'spectra' of A - i.e. the nature of eigenvalues.

1) λ 's are discrete $\rightarrow A|v_i\rangle = \lambda_i|v_i\rangle$ as before $\{|v_i\rangle\}_{i=1 \dots n}$ - n could be finite or infinite

2) In addition there could be a continuous spectrum

of eigenvalues associated to a basis $\langle v_\lambda |$:-

$$\forall |w\rangle \in V$$

$$\langle v_\lambda | A | w \rangle = \lambda \langle v_\lambda | A | w \rangle; \quad \lambda \in \mathbb{R}$$

N.B. This is not same statement as $A|w\rangle = \lambda|w\rangle$!

Here A is a self-adjoint operator to be thought of as acting on $\langle v_\lambda |$

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Example: Consider momentum operator again

$$P = -i\hbar \frac{d}{dx} . \quad \text{Trying to solve } P|\Psi\rangle = \lambda |\Psi\rangle$$

$|\Psi\rangle \leftarrow e^{i\lambda x/\hbar}$ — plane wave solutions of Schrödinger

equation. But $\Psi(x) \sim e^{i\lambda x/\hbar}$ are not normalizable

so $\in L^2(-\infty, \infty)$. So we cannot identify $|\Psi\rangle$

above with a ket. $|\Psi\rangle \in L^2$. But we can

associate it to a bra vector $\langle v_2 | \leftrightarrow e^{i\lambda x/\hbar}$

$$\begin{aligned} \langle v_2 | P |\Psi\rangle &= \int_{-\infty}^{\infty} (e^{i\lambda x/\hbar})^* (-i\hbar \frac{d}{dx} \Psi(x)) dx \\ &= \lambda \langle v_2 | \Psi \rangle \end{aligned}$$

The rhs exists for $\Psi \in L^2$ — we recognize it as

related to continuous Fourier transform of $\Psi(x) \rightarrow \Psi(p)$

$$p \sim \lambda/\hbar$$

See (12)^b →

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Example of momentum Eigenfunctions

which have finite norm

Consider 1-d system where have periodic box

$$\Psi(x) = \Psi(x+2\pi R) \quad \text{basis: } \Psi_n(x) \sim A e^{inx/R} \sim |\Psi_n\rangle.$$

Can choose A s.t. $\langle \Psi_n | \Psi_n \rangle = 1$.

$$\langle X | \Psi \rangle = \int_0^{2\pi R} X^*(x) \Psi(x) dx.$$

$$\rightarrow \langle \Psi_n | \Psi_m \rangle = \delta_{nm}$$

$$\hat{P} |\Psi_n\rangle \Leftrightarrow -i\hbar \frac{d}{dx} A e^{inx/R} = \left(\frac{n\hbar}{R}\right) A e^{inx/R}.$$

So $|\Psi_n\rangle$ are eigenfunctions of \hat{P} .

In this case $\hat{P} : V \rightarrow V$ $V =$ space of finite norm periodic functions in \mathbb{C} .

\hat{P} is self-adjoint

$$(\Rightarrow \text{also Hermitian}) \quad (\hat{P})^+ = \hat{P}$$

$$V^* = V$$

Finally, whilst it is true in the above example that there is no ket associated to $\langle U_2 |$.

we can in fact associate a ket if instead of $\langle U_2 |$ we consider superposition $\int d\lambda c(\lambda) \langle U_2 |$ with $c(\lambda)$ some coeffs.

The corresponding ket is $\int d\lambda c(\lambda) |U_2\rangle$.

- or in our example $\int d\lambda c(\lambda) e^{i\lambda x/\hbar} \leftrightarrow |U_2\rangle$

By putting conditions on $c(\lambda)$ we can demand

$|U_2\rangle \in L^2$ even though eigenfunction $e^{i\lambda x/\hbar} \notin L^2$.

- E.g. take $c(\lambda) \sim e^{-\lambda^2/a^2}$ (Gaussian)

then $|U_2\rangle \sim \int d\lambda e^{-\lambda^2/a^2} e^{i\lambda x/\hbar} \sim e^{-a^2 x^2}$

- a Gaussian in x -space and $\in L^2$!

By an abuse of notation we refer to these

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By an abuse of notation these kets

built from $\int d\lambda c(\lambda) \langle U_\lambda |$ are still referred to

as " $|U_\lambda\rangle$ " - even though there is in fact $\int d\lambda c(\lambda)$

Then the completeness relation for the case
where both discrete and continuous eigenvalues of \hat{H}

$$\sum_i \underbrace{|U_i\rangle \langle U_i|}_{\text{discrete e-value/e-vector}} + \underbrace{\int d\lambda |U_\lambda\rangle \langle U_\lambda|}_{\text{continuous e-value/e-vector}} = \mathbb{I}$$

$|U_\lambda\rangle$

Ortho-normality : $\langle U_i | U_j \rangle = \delta_{ij}$

$$\langle U_\lambda | U_{\lambda'} \rangle = \delta(\lambda - \lambda')$$

The Postulates of Quantum Mechanics

(Revisited)

Let's write down the fundamental postulates of Q.M. making use of some of the mathematical structures we have discussed thus far:-

1. At a fixed time t_0 , the state of a system is defined as a vector $|4\rangle$ in a Hilbert space \mathcal{H} .
2. Every physical (measurable) quantity A is associated with a self-adjoint operator \hat{A} (called 'observable')
3. The result of a measurement of A is always one of the eigenvalues of the corresponding operator \hat{A} .
4. The probability of a measurement yielding the eigenvalue a is $\|P_a|4\rangle\|^2$ where $\langle 4|4\rangle = 1$ and P_a is projection operator on space of eigenvectors $|4_a\rangle$ having eigenvalue a : $A|4_a\rangle = a|4_a\rangle$
 $P_a|4\rangle = |4_a\rangle$

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5. After a measurement of A yielding the eigenvalue a the state of the system changes ('collapses') from $|4\rangle$ to $P_a|4\rangle / \|P_a|4\rangle\|$.

6. The time evolution of the system is

described $\hat{H}|4(t)\rangle = i\hbar \frac{d}{dt}|4(t)\rangle$.

where \hat{H} is Hamiltonian operator.

Examples:

Consider a Hilbert space \mathbb{C}^3 and $|\psi\rangle \in \mathbb{C}^3$. Let observable A be represented as the 3×3 matrix:-

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and consider } |\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

to represent state of system at some fixed time.

It can be shown that A has 3 eigenvectors (normalized to unity):-

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$$|4_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |4_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |4_3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- These are also orthogonal (as follows from previously mentioned theorem concerning eigenvectors of Hermitian matrices)

$$\langle 4_i | 4_j \rangle = \delta_{ij}$$

$|4_2\rangle$, has e-value $\neq 2$, $|4_1\rangle, |4_3\rangle$ have e-value $+2$.

$$\text{Using fact that } \mathbb{I} = \sum_{i=1}^3 |4_i\rangle \langle 4_i| = P_{(2)} + P_{(-2)}$$

$$P_{(2)} = \text{projector operator onto } +2 \text{ eigenspace} = |4_1\rangle \langle 4_1| + |4_3\rangle \langle 4_3|$$

$$P_{(-2)} = \dots = |4_2\rangle \langle 4_2|.$$

$$\text{So } |\psi\rangle = \sum_{i=1}^3 \langle 4_i | \psi \rangle |4_i\rangle = P_{(2)} |\psi\rangle + P_{(-2)} |\psi\rangle.$$

Probability of e.g. measuring e-value -2 is $\|P_{(-2)}|\psi\rangle\|^2$

$$= \langle \psi | P_{(-2)} P_{(2)} |\psi\rangle = \langle \psi | P_{(-2)} |\psi\rangle$$

$$P_{(-2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} (1 - 0) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\text{So } \langle \phi | P_{(-2)} | \phi \rangle = \left(\frac{1}{\sqrt{2}} \right)^2 (-i|01\rangle) \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4} (-i|01\rangle) \begin{pmatrix} i \\ -i \\ 0 \end{pmatrix} = +\frac{1}{4} \cdot //$$

So probability of measuring e-value -2 is $\frac{1}{2}$.

Similarly, probability of measuring e-value +2 is -

$$\| P_{(2)} | \phi \rangle \| ^2 = \langle \phi | P_{(2)} P_{(2)} | \phi \rangle = \langle \phi | P_{(2)} | \phi \rangle$$

$$P_{(2)} = |\Psi_1\rangle \langle \Psi_1| + |\Psi_3\rangle \langle \Psi_3|$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \otimes (110) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes (001)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\| P_{(2)} | \phi \rangle \| ^2 = \frac{1}{2} (-i|01\rangle) \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4} (-i|01\rangle) \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$$

$$= \frac{3}{4} //$$