

QMS Week 12

NOT FOR  
EXAMINATION

①

## Identical Particles in Quantum Mechanics

Here we will discuss an unusual 'problem' that is apparent when we consider multiple identical particles in Q.M.

Before doing so, let us think about identical particles in classical physics. The particles would have identical properties such as mass, charge, ... but in principle we can imagine labelling them so that we can keep track of them during their motion (e.g. 2 particles one could be 'red' the other 'blue' etc.).

We could then, in principle, follow each particle as it moves along its classical trajectory.

So in some sense, even though the 2 particles ②  
have identical properties, we can 'distinguish' between them  
in principle (in practice it may be hard because of collisions  
etc..)

In the Quantum theory things are very different!

Here identical particles are truly indistinguishable, even  
in theory.

The point is, as we have learned, the most 'information'  
we can simultaneously know about a particle are the  
eigenvalues associated to a maximum set of commuting  
variables. It is these and only these that we can use  
to 'label' our particles.

Consider a 2 particle system. Let one particle  
be characterized by the ket  $|K'\rangle$  where here  $K'$

is a generalized label that refers to the aforementioned maximal commuting set of observables and their eigenvalues. Similarly, let  $|k''\rangle$  represent the second identical particle.

The state corresponding to the 2-particles can be

expressed:- 
$$|\Psi\rangle = |k'\rangle \otimes |k''\rangle$$

$\nearrow$   
particle 1

$\swarrow$   
particle 2

and the Hilbert space  $\mathcal{H}_{TOT} = \mathcal{H} \otimes \mathcal{H}$ , where  $\mathcal{H}$  is the Hilbert space of each single particle.

However, we could also consider a 'new' state

$$|\Psi'\rangle = |k''\rangle \otimes |k'\rangle$$

where the 'labels'  $k''$ ,  $k'$  have been permuted.

This is also <sup>seemingly</sup> a perfectly good descriptor of the 2 particles, because they are indistinguishable.

[ Note: this would not be the case if the particles were not identical. ]

Recall that from definition of tensor product,

$$|k'\rangle \otimes |k''\rangle \neq |k''\rangle \otimes |k'\rangle \text{ in general.}$$

$$\text{[ e.g. consider } |k'\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |k''\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; |k'\rangle \otimes |k''\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{whereas } |k''\rangle \otimes |k'\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ]$$

so they are different states, apparently describing same

2 particles! Indeed there is a class of equivalent

$$\text{states } |\tilde{\Psi}\rangle \equiv c_1 |k'\rangle \otimes |k''\rangle + c_2 |k''\rangle \otimes |k'\rangle$$

$$|c_1|^2 + |c_2|^2 = 1$$

Making a measurement of observable and finding  $k'$  for

one particle and  $k''$  the other holds not just for

$$|\Psi\rangle, |\Psi'\rangle \text{ but also } |\tilde{\Psi}\rangle.$$

This demonstrates the notion of 'exchange degeneracy' when we consider identical particles in Q.M.

The situation as it stands is unsatisfactory since new parameters  $c_1, c_2$  have been introduced. For systems of many identical particles, the number of coefficients  $c_i$  increases so for  $N$  particles we have  $N$  complex numbers  $\{c_i, i=1 \dots N\}$  with only 1 condition  $\sum_{i=1}^N |c_i|^2 = 1$

There is a very intriguing 'solution' to this degeneracy which has been supported by all experimental tests to date and which has profound implications for the structure of all matter.

Let us define a permutation operator  $\hat{P}_{12}$

(6)

$$\hat{P}_{12} |k'\rangle \otimes |k''\rangle = |k''\rangle \otimes |k'\rangle$$

We may also define  $\hat{P}_{21}$  but it's identical to  $\hat{P}_{12}$

$$\text{So } \hat{P}_{12} = \hat{P}_{21} \quad ; \quad \hat{P}_{12}^2 = \hat{I}$$

Now suppose we look at a simplified case where there is only 1 operator whose eigenvalues specify the kets  $|k'\rangle, |k''\rangle$ . Call this  $\hat{A}$ , and let  $\hat{A}_1, \hat{A}_2$  be its action on particle 1 and 2 states:-

$$\hat{A}_1 |k'\rangle \otimes |k''\rangle = k' |k'\rangle \otimes |k''\rangle$$

$$\hat{A}_2 |k'\rangle \otimes |k''\rangle = k'' |k'\rangle \otimes |k''\rangle$$

$$\begin{aligned} \text{Now } \hat{P}_{12} \hat{A}_1 \hat{P}_{12}^{-1} \hat{P}_{12} |k'\rangle \otimes |k''\rangle &= k' \hat{P}_{12} |k'\rangle \otimes |k''\rangle \\ &= k' |k''\rangle \otimes |k'\rangle \\ &= (\hat{P}_{12} \hat{A}_1 \hat{P}_{12}^{-1}) |k''\rangle \otimes |k'\rangle \end{aligned}$$

Now swap  $k' \leftrightarrow k''$  we find a consistency

condition that

$$\hat{P}_{12} \hat{A}_1 \hat{P}_{12}^{-1} = \hat{A}_2$$

This shows there is a natural action of  $\hat{P}_{12}$  on observable operators which exchanges  $1 \leftrightarrow 2$ .

Consider now the Hamiltonian  $\hat{H}$  of 2 identical particles. In terms of individual 1, 2 labels on e.g. position  $\vec{x}_1, \vec{x}_2$  and momenta  $\vec{p}_1, \vec{p}_2$  it must be

symmetric:

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \underbrace{V_{\text{int}}(|\vec{x}_1 - \vec{x}_2|)}_{\text{mutual interaction}}$$

$$+ \underbrace{V_{\text{ext}}(\vec{x}_1)}_{\text{interaction with external fields}} + \underbrace{V_{\text{ext}}(\vec{x}_2)}_{\text{interaction with external fields}}$$

$$\Rightarrow \hat{P}_{12} \hat{H} \hat{P}_{12}^{-1} = \hat{H}$$

$$\therefore \boxed{[\hat{P}_{12}, \hat{H}] = 0}$$

This last result has very important consequences:

Let us imagine that we consider only those states  $|\psi_+\rangle$ ,  $|\psi_-\rangle$  in our 2-identical particle system that are eigenstates of  $\hat{P}_{12}$ :

$$\hat{P}_{12} |\psi_+\rangle = |\psi_+\rangle \quad ; \quad \hat{P}_{12} |\psi_-\rangle = -|\psi_-\rangle$$

The eigenvalues of  $\hat{P}_{12}$  are clearly just  $+1$ , or  $-1$

Since  $\hat{P}_{12}^2 = \hat{I}$ .

Because  $[\hat{H}, \hat{P}_{12}] = 0$ , it then

follows that  $[\hat{U}(t-t_0), \hat{P}_{12}] = 0$ , where

$\hat{U} = e^{-i\hat{H}(t-t_0)/\hbar}$  is the evolution operator.

(9)

This means that a state which is an eigenstate of  $\hat{P}_{12}$  at  $t = t_0$  remains in that eigenstate at all later times.

→ see (p10) for  $|\psi_{\pm}\rangle$ .

This is how 'nature' gets around the problem of exchange degeneracy! In all experiments to date,

we find that identical particles are described either by symmetric states or anti-symmetric states under exchange of any two particles.

Identical particles with symmetric states are also found to have integer spin (including 0) and are called BOSONS.

Identical particles found in anti-symmetric states have odd integer  $\times \frac{1}{2}$  spin - are called FERMIONS

POSTULATE OF SPIN-STATISTICS

It's clear that in terms of the states  $|k'\rangle$  and  $|k''\rangle$ , the  $\hat{P}_{12}$  eigenstates  $|\psi_+\rangle$ ,  $|\psi_-\rangle$  are:-

$$|\psi_+\rangle = \frac{1}{\sqrt{2}} (|k'\rangle \otimes |k''\rangle + |k''\rangle \otimes |k'\rangle)$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}} (|k'\rangle \otimes |k''\rangle - |k''\rangle \otimes |k'\rangle)$$

with  $\langle \psi_+ | \psi_+ \rangle = 1$  ;  $\langle \psi_- | \psi_- \rangle = 1$

and  $\langle \psi_+ | \psi_- \rangle = \langle \psi_- | \psi_+ \rangle = 0$ .

$\frac{1}{\sqrt{2}}$  is chosen to have unit norm and we assume

$k' \neq k''$  so that  $\langle k' | k'' \rangle = 0$  for 1-particle states.

(note if  $k' = k''$ ,  $|\psi_-\rangle \equiv 0$  and obviously only symmetric combination is possible).

We can define the symmetrizer / anti-symmetrizer operators  $\hat{S}_{12}$ ,  $\hat{A}_{12}$  as:-

$$\hat{S}_{12} \equiv \frac{1}{2}(\hat{I} + \hat{P}_{12}) \quad ; \quad \hat{A}_{12} \equiv \frac{1}{2}(\hat{I} - \hat{P}_{12})$$

Then considering the general 2 particle state

$$c_1 |k'\rangle \otimes |k''\rangle + c_2 |k''\rangle \otimes |k'\rangle$$

we considered earlier, and acting with  $\hat{S}_{12}$ ,  $\hat{A}_{12}$  :-

$$\begin{aligned} &\hat{S}_{12} (c_1 |k'\rangle \otimes |k''\rangle + c_2 |k''\rangle \otimes |k'\rangle) \\ &= \frac{c_1 + c_2}{2} (|k'\rangle \otimes |k''\rangle + |k''\rangle \otimes |k'\rangle) \\ &\sim |Y_+\rangle \end{aligned}$$

$$\text{and } \hat{A}_{12} (c_1 |k'\rangle \otimes |k''\rangle + c_2 |k''\rangle \otimes |k'\rangle)$$

$$\begin{aligned} &= \frac{c_1 + c_2}{2} (|k'\rangle \otimes |k''\rangle - |k''\rangle \otimes |k'\rangle) \\ &\sim |Y_-\rangle \end{aligned}$$

Thus applying the spin-statistics postulate we discussed earlier, we see there is a unique state (once we fix normalization) describing the 2-identical particle state and this solves the exchange degeneracy problem.

Example:

Let's consider case of 2 identical spin  $\frac{1}{2}$  particles.

We have seen in previous lectures that the total

spin Hilbert space  $H_{\text{TOT spin}} = H_{\frac{1}{2}} \otimes H_{\frac{1}{2}}$  where  $H_{\frac{1}{2}}$  is

the 2-dimensional Hilbert space of  $j = \frac{1}{2}$  single particle state.

From previous lectures on addition of angular momentum, we have

seen that  $H$  describes a reducible representation of  $SO(3)$

rotation group, and is of dimension  $2 \times 2 = 4$ .

We saw the states on  $H$  can be expressed as  $j=1, j=0$  irreducible representations:-

$$|X_{j=1}\rangle = \{ |++\rangle, |--\rangle, \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \}$$

$$|X_{j=0}\rangle = \{ \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \}$$

where we use shorthand notation  $m_j = +\frac{\hbar}{2} = '+'$ ,

$m_j = -\frac{\hbar}{2} = '-'$ ,  $|m_j, m_{j'}\rangle \equiv |m_j\rangle \otimes |m_{j'}\rangle$ .

Notice that states  $|X_{j=1}\rangle$  are symmetric under exchange of 2 particle z-component of spin whereas  $|X_{j=0}\rangle$  is anti-symmetric.

If we consider total wavefunction of 2-particle states, they will take form:-

$$\Psi_{j=1}(\vec{x}_1, \vec{x}_2) = \phi_{j=1}(\vec{x}_1, \vec{x}_2) |X_{j=1}\rangle$$

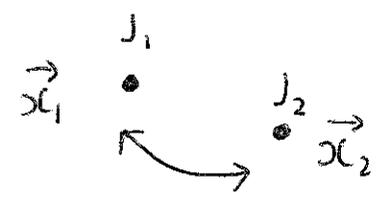
$$\Psi_{j=0}(\vec{x}_1, \vec{x}_2) = \phi_{j=0}(\vec{x}_1, \vec{x}_2) |X_{j=0}\rangle$$

By the spin-statistics postulate, we require

$$\Psi_{j=1}(\vec{x}_1, \vec{x}_2) \text{ and } \Psi_{j=0}(\vec{x}_1, \vec{x}_2) \text{ be anti-symmetric}$$

under exchange not only of spins of particle 1, 2  
but at the same time, exchange of positions  $\vec{x}_1, \vec{x}_2$

Since  $|X_{j=1}\rangle$  is symmetric and  $|X_{j=0}\rangle$  anti-symmetric under spin exchanges  $\Rightarrow$



$$\Phi_{j=1}(\vec{x}_2, \vec{x}_1) = -\Phi_{j=1}(\vec{x}_1, \vec{x}_2)$$

and

$$\Phi_{j=0}(\vec{x}_2, \vec{x}_1) = +\Phi_{j=0}(\vec{x}_1, \vec{x}_2)$$

$|\Phi_{j=1}(\vec{x}_1, \vec{x}_2)|^2 d^3x_1 d^3x_2$  = probability that  
a measurement of finding particle 1 in volume  $d^3x_1$   
at  $\vec{x}_1$  and particle 2 in volume  $d^3x_2$  at  $\vec{x}_2$ ,  
for the  $j=1$  state, with a similar definition for the  
 $j=0$  state.

Notice that for  $j=1$  state,  $\phi_{j=1}(\vec{x}_1, \vec{x}_2) \rightarrow 0$

as  $\vec{x}_1 \rightarrow \vec{x}_2$ . We also note that it is only in

the  $j=1$  state that particles 1 and 2 can occupy same spin states (i.e.  $1++$ ,  $1--$  states).

In the  $j=0$  case,  $\phi_{j=0}(\vec{x}_1, \vec{x}_2)$  need not

vanish as  $\vec{x}_1 \rightarrow \vec{x}_2$  BUT in this state the

2 particles do not occupy same spin states.

The net result of the above observations is

the famous Pauli Exclusion principle :

that no 2 identical fermions (spin  $\frac{1}{2}$  in this case e.g. electrons, quarks, neutrinos etc.) can occupy the

same quantum states.

We can extend these ideas to systems of  $N$  identical particles. In this case the wave-functions  $\Psi(\vec{x}_1, \dots, \vec{x}_N)$  depend on positions  $\vec{x}_1, \dots, \vec{x}_N$  of the particles, and may also include spin degrees of freedom. Then, applying spin statistics rule.

if we have  $N$  identical bosons,

$$\hat{P}_{ij} \Psi(\vec{x}_1, \dots, \vec{x}_N) = + \Psi(\vec{x}_1, \dots, \vec{x}_N) \quad \begin{matrix} i \neq j \\ i, j = 1, \dots, N \end{matrix}$$

while for  $N$  identical fermions we have:-

$$\hat{P}_{ij} \Psi(\vec{x}_1, \dots, \vec{x}_N) = - \Psi(\vec{x}_1, \dots, \vec{x}_N) \quad \begin{matrix} i \neq j \\ i, j = 1, \dots, N \end{matrix}$$

here  $\hat{P}_{ij}$  is the operator that exchanges  $i^m$  and  $j^m$

particles, which means  $\vec{x}_i \leftrightarrow \vec{x}_j$  but also

the exchange of  $(j, m_j)$  quantum numbers.

