

Week 1: Groups & Vector SpacesSome Useful algebraic structuresGroups

A group is a set of elements G together with a group 'operation' \circ that tells us how to combine any two elements, to give another. The 4 axioms that define a group are:-

- 1) Closure: $\forall a, b \in G, a \circ b \in G$.
- 2) Associativity: $\forall a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$
- 3) Identity element: $\exists e \in G$ s.t. $\forall a \in G, a \circ e = e \circ a = a$.
- 4) Existence of inverse element: $\forall a \in G, \exists b \in G$
s.t. $a \circ b = b \circ a = e$

As we shall see in the course, groups allow us to understand symmetries present in any physical

(2)

System because such symmetries form a 'representation' of a group. The properties/nature of the group will depend on the particular symmetry in question; the main idea is that groups allow for a systematic study of symmetries.

Such symmetries are present in 'classical' systems and are often very helpful in solving e.g. dynamical problems (for example exploiting rotational invariance in physical systems allows for great simplification in solving certain problems). Likewise in the quantum theory, if these symmetries are present they are 'realized' in a different way compared to the classical version of the theory.

However that apart, the existence of such symmetries is equally important in 'solving' the underlying quantum mechanics. As we shall see this is true even if our system does not have an exact symmetry but only one which is approximate in that e.g. certain ('small') terms in the Hamiltonian break the symmetry. This will lead to the development of perturbation theory techniques.

Some simple examples of Groups

Ex 1:

Consider $G = \mathbb{Z} = \{\text{set of integers}\} = \{0, \pm 1, \pm 2, \dots\}$.

and define composition rules 'o' = ^{standard} addition of numbers.

Then $\{\mathbb{Z}, +\}$ defines a group.

check:

- 1) Let a, b be any $\in \mathbb{Z}$. $a \circ b = a + b$ - clearly this is also an integer and hence in \mathbb{Z} .
- 2) Given $a, b, c \in \mathbb{Z}$. obvious that $(a + b) + c = a + (b + c)$ because addition is associative.
- 3) The identity element is just '0' because then $e = '0'$; $a \circ e \equiv a + 0 = 0 + a \equiv e \circ a$ for any $a \in \mathbb{Z}$.
- 4) Given that $e \equiv '0'$, for any $a \in \mathbb{Z}$,
it's inverse is simply $b = -a$: $a \circ b = e \leftrightarrow a + (-a) = 0$

We also note another simple property of $\{\mathbb{Z}, +\}$

namely $a \circ b = b \circ a$ [$\leftrightarrow a + b = b + a$ under addition]

This condition is true $\forall a, b \in \mathbb{Z}$. It means the group

$\{\mathbb{Z}, +\}$ is Abelian, (i.e. ^{the} group ^{is} commutative) (4)
composition \uparrow

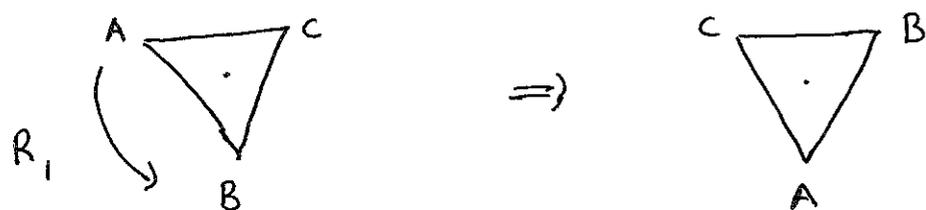
Dimension of group:

This is defined as the number of elements in G .

The above example is one where the dimension is infinite as there are infinitely many $\in \mathbb{Z}$.

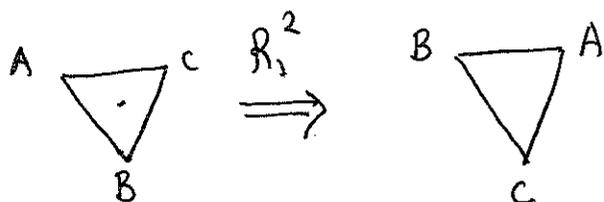
Ex 2.

Consider an equilateral triangle and rotations about its centre by 120° :-



Such a rotation leaves the triangle unchanged. If R_1 represents this rotation then

$R_1 \circ R_1 = R_1^2$ is clearly a rotation by 240°



Finally if we compose R_1 with itself 3

times : $R_1 \circ R_1 \circ R_1 \equiv R_1^3 = \text{rotation by } 360^\circ$.

But a rotation by $360^\circ = \text{same as } \overset{\text{not}}{\text{doing anything!}}$

$\therefore R_1^3 = e = \text{identity element}$.

So our group consists of 3 elements: $\{R_1, R_1^2, e\}$

It has dimension 3, is abelian (check)

and mathematically is equivalent to the finite group \mathbb{Z}_3

- the group of integers modulo 3. $[\mathbb{Z}_3 = \mathbb{Z} \text{ mod } 3]$.

That is any $a \in \mathbb{Z}$ is identified with $a \pm 3n, n \in \mathbb{Z}$ and they are not independent elements. Thus $\mathbb{Z}_3 = \{0, 1, 2\}, +\}$

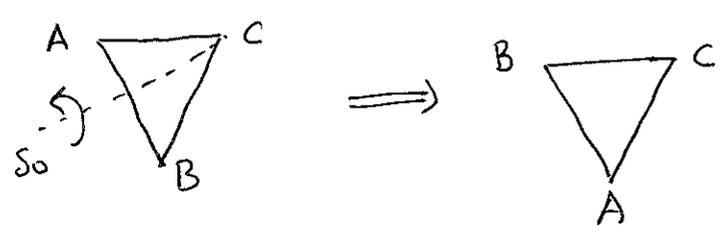
with elements like $-1 = 2 - 3 \equiv '2'$; $-2 = 1 - 3 \equiv '1'$ etc..]

Staying with the example of the equilateral triangle,

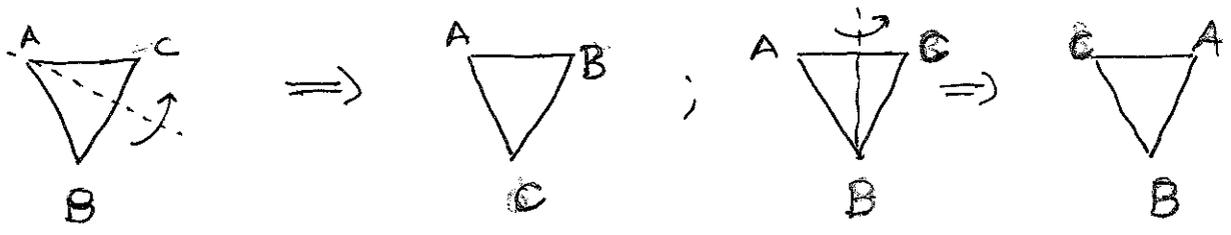
we note there are more than just 120° rotations that

leave the triangles unchanged - we can also

consider reflections: -



There are 2 similar reflections: -



Exercise: consider all possible combinations of elements

R_1 and S_0 and show they generate a group

This is called Dihedral group D_3 - it has dimension 6 and describes all symmetries of an equilateral triangle.

You will discover D_3 is not abelian, that is

$a \circ b \neq b \circ a$ in general. Eg. consider R_1 and S_0 :-

$$R_1 \circ S_0 \neq S_0 \circ R_1 \quad \left(\begin{array}{l} R_1 \circ S_0 : (ABC) \rightarrow \overset{CBA}{(A \ B \ C)} \\ S_0 \circ R_1 : (ABC) \rightarrow (ACB) \end{array} \right)$$

etc...

In fact D_3 is equivalent to the symmetric

group S_3 = set of all permutations of 3 objects:-

$$\left\{ \begin{array}{l} (ABC), (CAB), (BCA) \\ (BAC), (ACB), (CBA) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} e, R, R^2 \\ S_0, S_0R, R, S_0R \end{array} \right\}$$

(using shorthand notation

$$R^2 \equiv R \circ R, S_0R \equiv S_0 \circ R, \text{ etc...})$$

Sub-groups

Often a group will have various sub-groups

being defined as a subset of elements $H \subset G$

which form a 'closed' group H by themselves. By closure we mean that H satisfies all 4 group axioms.

Example: In the above group of symmetries of an equilateral triangle, D_3 (or equivalently S_3), we can notice that there is a \mathbb{Z}_3 subgroup generated by

$$R_1 = \{e, R, R^2\} \quad \text{Thus } \mathbb{Z}_3 \subset D_3$$

Also there is a \mathbb{Z}_2 subgroup generated by $S_0 = \{e, S_0\}$

In fact D_3 is itself generated from R_1 and S_0 as we have seen.

So far we have discussed groups and discrete symmetries. Such groups are often referred to as 'discrete' because there is no continuous parameter associated with them.

But there are a whole class of groups - called continuous groups (or Lie Groups) which do have elements that depend on continuous parameters. An obvious example are the rotations of a 2-sphere about any axis passing through its centre. The rotation angles can be arbitrary - not just discrete values as we saw in the equilateral triangle case.

The corresponding group is the 3d rotation group which is a non-abelian group that we will discuss in greater detail later. Physics is full of examples where continuous groups play an important role. Perhaps most famous are those associated with Strong Nuclear, Weak Nuclear & electromagnetic forces of the standard model of particle physics.

In this case the symmetries in question are not physical rotations in 3d - but rather 'internal' symmetries.....

Later on in the course we will meet other examples (parity, time reversal, translation etc...)

Vector Spaces

The concept of a vector space is another area of linear algebra that has profound importance in almost every area of physics we care to think of. Obvious examples are in Newtonian Mechanics and its extension to space-time vectors of Special Relativity, with vectors representing fundamental quantities such as position, momentum, angular momentum, etc... Quantum Mechanics also at a fundamental level is a theory that can be expressed in terms of vector spaces and the linear maps that act on them.

Definition: A real vector space V is an abelian group under vector addition and furthermore each $v \in V$ can be scaled by any real number $a \in \mathbb{R}$. That is if $v \in V$

$\Rightarrow av \in V$. Multiplication by scalar a must satisfy following conditions:-

1) Distributivity: (wrt vector addition) $\forall a \in \mathbb{R}$ and $\forall u, v \in V$.

$$a(u+v) = au + av$$

2) Distributivity w.r.t real number addition:

$$\forall a, b \in \mathbb{R}, \forall v \in V \quad (a+b)v = av + bv.$$

3) $\forall a, b \in \mathbb{R}, \forall v \in V, \quad a(bv) = (ab)v.$

4) Multiplication by identity and zero: $\left. \begin{matrix} 1 \cdot v = v \\ 0 \cdot v = 0 \end{matrix} \right\} \forall v \in V$

A complex vector space is simply the extension where we allow rescaling of vectors by complex numbers $a \in \mathbb{C}$ rather than $a \in \mathbb{R}$.

The concept of linearly independent vectors is central to vector spaces. A set of vectors $\{v_1, v_2, \dots, v_n\}$ are said to

be linearly independent if

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

has ^{as the} only solution $a_1 = a_2 = \dots = a_n = 0.$ ($a_i \in \mathbb{R}$ or $a_i \in \mathbb{C}$)

A corollary of this is that if a vector space V has a maximum of n such linearly independent vectors, dimension of $V = n$. Such a set of vectors define a basis for V .

Given that a basis of an n -dimensional vector space V can be formed by n linearly independent vectors $\{v_1, \dots, v_n\}$ it follows that V is isomorphic (in 1-1 correspondence) with elements of \mathbb{R}^n (or \mathbb{C}^n for complex V). This is because we can always decompose

any $v \in V$ in terms of linear combinations of $\{v_i, i=1 \dots n\}$:-

$$v = \sum_{i=1}^n a_i v_i$$

The set of $\{a_i\}$ can be thought of as an n -tuple in \mathbb{R}^n

$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \leftrightarrow$ column vector. Moreover for given v , $\{a_i\}$

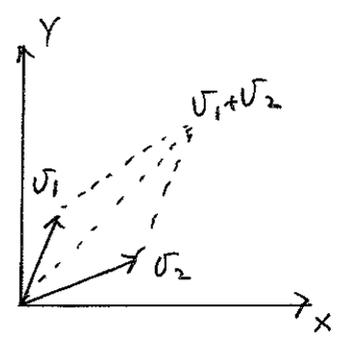
are unique since if $v = \sum_{i=1}^n b_i v_i \Rightarrow \sum_{i=1}^n (a_i - b_i) v_i = 0$

but then $a_i = b_i \forall i$ as $\{v_i\}$ are linearly independent.

There are many 'obvious' examples of a vector space.

E.g. the Euclidean plane -

with e.g. basis given by e_1, e_2



also the complex plane \mathbb{C}^2 , and obvious extensions to higher dimensions.

Other examples perhaps not so familiar involve ^{spaces of} algebraic functions. E.g. consider space of polynomials ^{in x} with real coefficients of degree 1, that is V consists of functions of form $a+bx$ for $a, b \in \mathbb{R}$. It's easy to check that this satisfies axioms of a vector space where 'vector addition' is standard addition e.g. $v_1 = a_1 + b_1x$, $v_2 = a_2 + b_2x$, $v_1 + v_2 = (a_1 + a_2) + (b_1 + b_2)x \in V$ etc...

Scalar Products & Hilbert Spaces

In physics, the vector spaces that we are interested in have an additional structure beyond just axioms above - they also possess a scalar (often called 'inner') product.

Inner product for a real vector space V is a bilinear map $V \times V \rightarrow \mathbb{R}$ ($V \times V \rightarrow \mathbb{C}$ for complex V)

i.e. for any $v_1, v_2 \in V$
 $(v_1, v_2) \in \mathbb{R}$ is the image of this map in \mathbb{R} .

The product (v_1, v_2) must satisfy following

conditions:-

1) Linearity: $\forall a_i \in \mathbb{C}, \forall w, v_i \in V, i=1,2$

$$(w, a_1 v_1 + a_2 v_2) = a_1 (w, v_1) + a_2 (w, v_2)$$

$$\text{and } (a_1 v_1 + a_2 v_2, w) = \bar{a}_1 (v_1, w) + \bar{a}_2 (v_2, w)$$

2) Conjugation symmetry: $\forall w, v \in V; (w, v) = \overline{(v, w)}$

(for real vector space, $\bar{a}_i = a_i$ and $\overline{(v, w)} = (v, w)$).

Thus 2) implies inner product is symmetric in this case.

Now 2) $\Rightarrow (v, v) \in \mathbb{R}$ and we additionally require

$$(v, v) > 0 \quad \forall v \neq 0.$$

This latter condition means scalar (or inner) product is positive definite. (v, v) is also often written as $\|v\|^2$ -

the norm of v . In Quantum Mechanics we will always require positive definite norms as here the vectors v are associated to states of the system as we shall see.

In contrast to this, the vectors that appear in e.g.

Schwarz inequality.

For any two vectors $v_1, v_2 \in$ Hilbert Space

$$|(v_1, v_2)|^2 \leq \|v_1\|^2 \|v_2\|^2$$

proof: consider $w = v_1 + av_2$ $a \in \mathbb{C}$

$$\|w\|^2 > 0$$

$$0 \leq \|w\|^2 = (v_1, v_1) + |a|^2 (v_2, v_2) \\ + a(v_1, v_2) + \bar{a}(v_2, v_1)$$

now choose $a = -\frac{(v_2, v_1)}{(v_2, v_2)}$ then

$$0 \leq (v_1, v_1) + \frac{|(v_2, v_1)|^2}{(v_2, v_2)} - \frac{|(v_2, v_2)|^2}{(v_2, v_2)} - \frac{|(v_1, v_2)|^2}{(v_2, v_2)}$$

$$\Rightarrow |(v_1, v_2)|^2 \leq \|v_1\|^2 \|v_2\|^2 //$$

Minkowski Space of Special Relativity do not

have a positive definite norm, rather $\|U\|^2 = U^\mu U^\nu \eta_{\mu\nu}$

where $\eta_{\mu\nu} = \text{diag}(-+++)$ as required by Lorentz invariance.

Some consequences of inner product:-

1) If $(v, w) = 0$ for $v, w \in V$, then v and w are said to be orthogonal.

2) A particular basis for V is called orthogonal

if $(v_i, v_j) = 0$ for $i \neq j$. If it is also the

case that $(v_i, v_i) = 1, \forall i$ then such a basis is

said to be orthonormal, so that $(v_i, v_j) = \delta_{ij}$.

3) For any vector $v \in V$ we have seen that

$$v = \sum_{i=1}^n a_i v_i. \quad \text{If } \{v_i\} \text{ is orthonormal then it}$$

follows that $a_i = (v_i, v)$

Schwarz inequality (see p14a)

Hilbert Space.

Roughly speaking a Hilbert Space is a vector space with positive definite inner product. Technically there is

an additional requirement that the vector space V be 'Cauchy complete'. We won't discuss what

this means in detail here (see p 7 of notes for a discussion)

other than to say this involves notion that

elements of V can be arbitrarily close to one another -

which gives rise to notion of convergence - e.g.

\mathbb{R} is an example of a Hilbert space (with $(,)$ defined by usual multiplication of numbers), and any real number $a \in \mathbb{R}$ can be thought of as limit of a sequence of near-by numbers

$(a_0, a_1, a_2, \dots, a_n)$ where $\lim_{n \rightarrow \infty} a_n = a$.

Likewise \mathbb{R}^n and \mathbb{C}^n are also examples with $(u, v) \equiv \sum_{i=1}^n u_i v_i$ ($\sum_{i=1}^n \bar{u}_i v_i$ for \mathbb{C}^n).

The problem arises when we consider $n \rightarrow \infty$ - i.e. infinite dimensional Hilbert spaces. These occur in QM!

We

In this situation we have to take care about convergence when dealing with infinite sums.

In particular a natural condition to impose is that the norm of any $v \in V$ is finite:-

$$\|v\|^2 \equiv \sum_{i=1}^{\infty} \bar{v}_i v_i < \infty \quad (\text{this is } \mathbb{C}^{\infty} \text{ case}) \quad (1)$$

it's a Hilbert space called ℓ^2

and so the notion of convergence becomes important.

Formally, Cauchy completeness requires the condition that

$$\text{given 2 sequences of vectors } v_m, v_n, \quad \|v_m - v_n\| \text{ can } \quad (2)$$

be made arbitrarily small for sufficiently large m, n ,

and that $\lim_{n \rightarrow \infty} v_n \in V$. [see Appendix for counter example]

Finally, we still have to have a concept of orthonormal

basis of the Hilbert space, even in the infinite dimensional case.

The importance of this is that we can then still find

$$\text{that :- } v = \sum_{i=1}^{\infty} a_i v_i \quad ; \quad a_i = (v_i, v) \quad i=1, 2, \dots$$

and $(v_i, v_j) = \delta_{ij}$

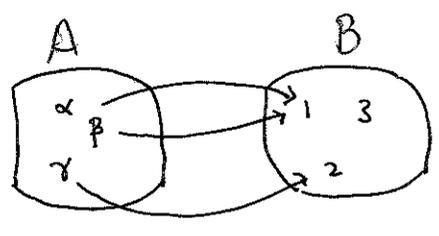
This condition is particularly important in Q.M. as

it is basis of the Expansion Theorem when v is

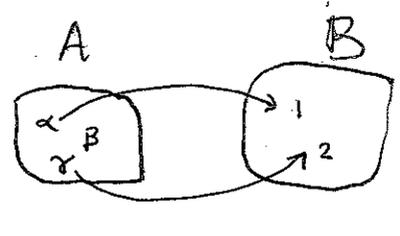
identified as a general state of the system.

Linear Maps between Vector Spaces:

Start with basic concept of a map between two sets A, B defined by a function $f: A \rightarrow B$. In order for f to be a function it must map each element in A to at most one element in B: $\forall a \in A, \exists b$ s.t $f(a) = b$. E.g. :-



is a function



is not a function.

Further conditions we can place on f:

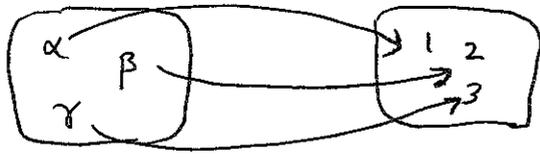
1) if all elements of B are images of some elements in A, the map is called 'onto' or surjective. $\forall b \in B, \exists a \in A$ s.t $f(a) = b$

2) If given two different elements in A, their images in B are also distinct, the mapping is called injective:

$$\forall a_1, a_2 \in A ; f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

A map that is both surjective & injective is called bijjective

eg.



In the case when A, B are just sets bijective \leftrightarrow isomorphism between A & B .

In cases where A, B have more structure (eg they are Vector spaces)

a bijective map is only an isomorphism if it preserves the linearity of A . This is why of all the mappings

that can act on Vector spaces, linear maps have a special place.

Linear functionals and Dirac Notation.

This is a good place to introduce the 'bra' and 'ket' notation first introduced by Paul Dirac, to represent elements of a Hilbert Space (or more generally a Vector space).

We will use this throughout the course.

An element of V is written as the 'ket' $|v\rangle$.

This could be an n -tuple (if $V = \mathbb{R}^n$ or \mathbb{C}^n), a polynomial of any degree or $\in l^2$ or $L^2 \dots$

The point is that one does not have to specify exactly how $|v\rangle$ is represented at this stage - which was partly the point of Dirac - that QM has an algebraic structure which is quite independent of how one realizes these structures e.g. for the purposes of calculation etc..

One can now consider a mapping from $V \rightarrow \mathbb{R}$ (or \mathbb{C} if V is complex) defined as a linear functional acting on V , (call it X .)

The linearity assumption implies:-

$$\forall |\psi_1\rangle, |\psi_2\rangle \in V, \forall a, b \in \mathbb{R} (\mathbb{C})$$

$$X(\underbrace{a|\psi_1\rangle + b|\psi_2\rangle}_{\in V}) = aX(|\psi_1\rangle) + bX(|\psi_2\rangle)$$

The set of all such maps X defines the dual space V^*

In Dirac notation, elements of V^* are denoted by the 'bra' vector $\langle X|$.

An important property of vector spaces with inner product (like the Hilbert spaces of Q.M.) is that the inner

product allows us to associate a bra vector

for any given ket. Consider $|\phi\rangle \in \mathcal{V}$. Then

taking the inner product of $|\phi\rangle$ with any other element (say $|\psi\rangle$) in \mathcal{V} defines a linear functional from $\mathcal{V} \rightarrow \mathbb{R}(\mathbb{C})$

$$(|\phi\rangle, |\psi\rangle) \equiv \langle \phi | \psi \rangle. \quad (\text{notice ordering here - because } \langle \phi | \psi \rangle \neq \langle \psi | \phi \rangle \text{ in general})$$

So the inner product is a 'bra-ket' in Dirac notation.

From properties that inner product satisfies (see earlier)

some properties of $\langle \phi | \chi \rangle$ can be derived:-

$$\langle \phi | \chi \rangle = \overline{\langle \chi | \phi \rangle}$$

$$\langle \phi | a\chi_1 + b\chi_2 \rangle = a\langle \phi | \chi_1 \rangle + b\langle \phi | \chi_2 \rangle$$

$$\langle a\phi_1 + b\phi_2 | \chi \rangle = \bar{a}\langle \phi_1 | \chi \rangle + \bar{b}\langle \phi_2 | \chi \rangle$$

$$\langle \chi | \chi \rangle > 0; \forall |\chi\rangle \neq 0$$

Notice that bra / ket vectors are related in an anti-linear

way; if ket is $(a|\psi_1\rangle + b|\psi_2\rangle) \Rightarrow$ bra is $(\bar{a}\langle \psi_1| + \bar{b}\langle \psi_2|)$

A natural question to ask is if there is always a 1-1 correspondence between bra and ket?

For finite dimensional case there is: proof is just take V to have orthonormal basis $\{|\psi_i\rangle, i=1 \dots n\}$.

Given the bra $\langle X|$ one can derive coefficients

$$\langle X|\psi_i\rangle = c_i.$$

$$\text{Then } \sum_{i=1}^n \bar{c}_i |\psi_i\rangle = \sum_{i=1}^n \overline{\langle X|\psi_i\rangle} |\psi_i\rangle \equiv |X\rangle$$

is the ket corresponding to the bra $\langle X|$.

In the infinite dimensional case there can be subtleties - because the \sum involve infinite series. Even if these converge - the resulting object may not be an element of V !

If V is a Hilbert space (stronger condition) - Cauchy completeness avoids this potential problem. But there are exceptions where we want to consider spaces V which are not quite Hilbert spaces - but 'almost'. An example will be given shortly.

Bra vectors for finite dimensional Hilbert space

Eg. Consider \mathbb{C}^3 . $|w\rangle \in \mathbb{C}^3$ just a 3 component column vector with complex entries: $|w\rangle \leftrightarrow \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

The standard inner product on \mathbb{C}^3 is just:

$$(|v\rangle, |w\rangle) \equiv (\bar{v}_1, \bar{v}_2, \bar{v}_3) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \sum_{i=1}^3 \bar{v}_i w_i$$

But $(|v\rangle, |w\rangle) \equiv \langle v|w\rangle \Rightarrow \langle v| \leftrightarrow (\bar{v}_1, \bar{v}_2, \bar{v}_3)$.
just h.c. of $|v\rangle$

Dirac's Delta

Considers space of complex function $f: \mathbb{R} \rightarrow \mathbb{C}$

such that $f(x)$ are analytic functions (infinitely differentiable)

and with $f(x) \rightarrow 0$ for $|x| \rightarrow \infty$ very rapidly (eg. $|x^m \frac{d^n f}{dx^n}| \rightarrow 0$

as $|x| \rightarrow \infty \forall n, m$)

This class of functions forms a vector space, call it V_f

with a scalar product we can choose as:-

$$(\mathcal{F}, f) \equiv \int_{-\infty}^{\infty} \bar{f}(x) f(x) dx.$$

(it's easy to check above satisfies properties required of an inner product)

Now consider the linear functional $f(x) \rightarrow f(0)$

This is a map $V_f \rightarrow \mathbb{C}$. Question - what is

'bra' vector corresponding to this?

well, $f(x=0) = \int_{-\infty}^{\infty} \delta(x) f(x) dx \equiv \langle \delta_0 | f \rangle$

$\underbrace{\hspace{10em}}_{(\delta, f)}$

If we define $|f\rangle \leftrightarrow f(x)$ then $\langle \delta_0 | \leftrightarrow \delta(x)$

But $\delta(x) \notin V_f$ (Dirac's delta function is not strictly a function but a distribution).

Therefore there is no element in V_f we can associate to $\langle \delta_0 |$ (i.e. a would-be " $|\delta_0\rangle$ ")

As a final remark, the space V_f turns out to be a little restrictive for applications in Q.M.

By relaxing the condition that $f \in V_f$ be arbitrarily differentiable, we arrive at simply requiring:-

$$\int_{-\infty}^{\infty} |g(x) f(x)| < \infty \quad ; \quad \text{this is a Hilbert Space.}$$

(called $L^2(-\infty, \infty)$)

Appendix : Cauchy Completeness

Here is a simple example of a space that isn't

Cauchy complete: Consider set of rational numbers

P/Q , $P, Q \in \mathbb{Z}$ (excluding 0). These form a vector space

if we restrict multiplication by $a \in \mathbb{Z}$ rather than $a \in \mathbb{R}$.

Consider the irrational number $\sqrt{2}$. This can be

expressed as:-

$$\sqrt{2} = 2 \left(1 - \frac{1}{2}\right)^{\frac{1}{2}} = 2 \lim_{N \rightarrow \infty} \left(1 - \sum_{i=1}^N \frac{c_i}{i!} \left(\frac{1}{2}\right)^i\right)$$

where we used binomial expansion of $\left(1 - \frac{1}{2}\right)^{\frac{1}{2}}$ and

$$c_1 = \frac{1}{2}, \quad c_{i>1} = \frac{1 \cdot 3 \cdot \dots \cdot 2i-1}{2^i}$$

Now call $\sum_{i=1}^n \frac{c_i}{i!} \left(\frac{1}{2}\right)^i = v_n$. $v_n \in \mathbb{V}$ (rational numbers)

because it is a finite sum of rational numbers.

It's easy to see sequence $\{v_n\}$ is a

because $\|v_n - v_m\| = |(v_n - v_m)|^{1/2}$ can be made as small

as we wish as long as we choose $n, m > N$ for

large enough N .

But $\lim_{n \rightarrow \infty} U_n = \left(1 - \frac{1}{\sqrt{2}}\right)$ which is clearly

irrational and so not an element of V !

Therefore the rational numbers are Cauchy incomplete, and not a Hilbert Space.

It is obvious how to make it complete -

just extend the rationals $\rightarrow \mathbb{R}$ - all the real numbers. This is a Hilbert Space.
