- Fermionic particles, whose wavefunction must be antisymmetric $|v\rangle_{F}=-P^{12}|v\rangle_{F}$.

Also the observables in composite systems with identical particles must have special properties under the exchange of the labels indicating the various particles. In particular, an observable $O$ must commute with the operation that permute the role of two particles $P^{i j}:\left[O, P^{i j}\right]=0, \forall i j$.

### 7.2.1 Examples and exercises.

Consider a system of two identical particles each one described by a state in $\mathbf{C}^{2}$. If this particles are boson then the possible states are

$$
\begin{align*}
\left|v_{1}\right\rangle_{B} & =\binom{1}{0} \otimes\binom{1}{0},\left|v_{3}\right\rangle_{B}=\binom{1}{0} \otimes\binom{1}{0}  \tag{7.6}\\
\left|v_{2}\right\rangle_{B} & =\frac{1}{\sqrt{2}}\left[\binom{1}{0} \otimes\binom{0}{1}+\binom{0}{1} \otimes\binom{1}{0}\right] .
\end{align*}
$$

On the other hand, if the two particles are fermions then there is only one possible state

$$
\begin{equation*}
|v\rangle_{F}=\frac{1}{\sqrt{2}}\left[\binom{1}{0} \otimes\binom{0}{1}-\binom{0}{1} \otimes\binom{1}{0}\right] . \tag{7.7}
\end{equation*}
$$

## Exercises.

- Consider a two particle state whose constituents are bosons. The initial state of the system is then described by a symmetric state. Is this property preserved by the time evolution? Why?


## 8 Symmetries.

We saw that Hermitean operators play a central role in Quantum Mechanics: they represent the observables of a physical system. There is another very important class of operators: the unitary operators $U$, that are the operators preserving the norm of any vectors

$$
\begin{equation*}
\left.\| U|\psi\rangle\left\|^{2}=\right\|\left\langle\psi \mid U^{\dagger} U \psi\right\rangle=\| \| \psi\right\rangle \|^{2}, \quad \forall|\psi\rangle . \tag{8.1}
\end{equation*}
$$

This implies that $U^{\dagger} U=1$. If we work with a finite dimensional Hilbert space, where we can represent the operators with matrices, then we can check explicitly if a matrix is unitary (see the example below). Unitary operators are important in QM, because they represent the action of a symmetry operation on a physical system: starting with a system
described by the ket $|\psi\rangle$, we can obtain the ket $U|\psi\rangle$ which describes the same system after the symmetry operation related to $U . U$ must be a unitary operator since we want to keep the normalization condition $\||\psi\rangle \|^{2}=1$.

It is easy to build explicitly unitary operators starting from the Hermitean ones we have been using so far. We can just repeat the trick we used in (5.1) with the Hamiltonian: if now $H$ is any operator satisfying $H=H^{\dagger}$, then we can repeat the same steps ${ }^{11}$ and prove that $U_{a}=\exp (\mathrm{i} a H)$ is a unitary operator for any real $a$.

### 8.1 Translations.

Consider a free particle in one dimension whose state is encoded by the wavefunction (5.14), which describes a particle around the position $x=0$ with a precision determined by $a$. Since the particle is free, we can displace it by $x_{0}$ (so that the Gaussian is centred in $x_{0}$ instead of $z_{0}$ ) and many observables, such as the energy of the system, should not change. The operator representing this operation should act on (5.14) as follows:

$$
\begin{equation*}
U_{T\left(x_{0}\right)}\left[\frac{A}{\sqrt{\pi} a} \mathrm{e}^{-\frac{x^{2}}{a^{2}}}\right]=\frac{A}{\sqrt{\pi} a} \mathrm{e}^{-\frac{\left(x-x_{0}\right)^{2}}{a^{2}}} . \tag{8.2}
\end{equation*}
$$

We can derive the form of $U_{T\left(x_{0}\right)}$ looking at the case of an infinitesimal translation (a very small $x_{0}$ ), so that we can Taylor-expand the right hand side of the above equation and keep only the first two terms

$$
\begin{equation*}
U_{T\left(x_{0}\right)}\left[\frac{A}{\sqrt{\pi} a} \mathrm{e}^{-\frac{x^{2}}{a^{2}}}\right]=\frac{A}{\sqrt{\pi} a} \mathrm{e}^{-\frac{x^{2}}{a^{2}}}+x_{0} \frac{d}{d x_{0}}\left[\frac{A}{\sqrt{\pi} a} \mathrm{e}^{-\frac{\left(x-x_{0}\right)^{2}}{a^{2}}}\right]_{x_{0}=0}+\ldots . \tag{8.3}
\end{equation*}
$$

From this result we see that $U_{T\left(x_{0}\right)}=1+x_{0} d / d x_{0}+\ldots=1-x_{0} d / d x+\ldots$. Thus, for small $x_{0}$ we can write $U_{T\left(x_{0}\right)}=1-\mathrm{i} x_{0} \hat{p} / \hbar+\ldots$ and by using the observation above we can readily guess that

$$
\begin{equation*}
U_{T\left(x_{0}\right)}=\exp \left(-\frac{\mathrm{i}}{\hbar} x_{0} \hat{p}\right) . \tag{8.4}
\end{equation*}
$$

At this point it is easy to see that all possible translation operators form a group (see the first Section):

$$
\begin{equation*}
U_{T\left(x_{0}\right)} U_{T\left(x_{1}\right)}=\exp \left(-\frac{\mathrm{i}}{\hbar} x_{0} \hat{p}\right) \exp \left(-\frac{\mathrm{i}}{\hbar} x_{1} \hat{p}\right)=\exp \left(-\frac{\mathrm{i}}{\hbar}\left(x_{0}+x_{1}\right) \hat{p}\right) \equiv U_{T\left(x_{0}+x_{1}\right)} . \tag{8.5}
\end{equation*}
$$

With these results we established an important fact

[^0]The momentum is directly related to the translation operation (technically speaking: the momentum is the generator of the translations).

Notice that, if an observable $O$ commutes with $\hat{p}$, then the results of a measurment of $O$ are the same if carried out on the system before or after a translation. For instance, in the case of a free particle we have $[H, \hat{p}]=0$; then the wavefunctions (5.14) and (8.2) yields the same results for a measurement of the energy.

### 8.1.1 Examples and exercises.

Notice that the result (8.4) is not tight to a particular realization of the position/momentum operators, such as the position-space wavefunction. For instance, when the translation operator acts on the momentum-space wavefunctions (see, for instance, Eq. (5.16)), then it simply multiplies $\psi(p)$ by the phase $\exp \left(-\mathrm{i} x_{0} p / \hbar\right)$.

## Exercises.

- Use the explicit form of the translation operator (8.4) and prove Eq. (8.2).


### 8.2 The rotations.

Another relevant group of symmetries is represented by the rotations (here we focus on the rotation in the 3 -dimensional space). The generator of the rotation is another very important observable: the angular momentum.

$$
\begin{align*}
L_{x} & =\hat{y} \hat{p}_{z}-\hat{z} \hat{p}_{y} \equiv L_{1},  \tag{8.6}\\
L_{y} & =\hat{z} \hat{p}_{x}-\hat{x} \hat{p}_{z} \equiv L_{2}, \\
L_{z} & =\hat{x} \hat{p}_{y}-\hat{y} \hat{p}_{x} \equiv L_{3} .
\end{align*}
$$

By using the canonical commutation relation $[\hat{x}, \hat{p}]=\mathrm{i} \hbar$, we obtain the following relations

$$
\begin{equation*}
\left[L_{1}, L_{2}\right]=\mathrm{i} \hbar L_{3} \quad, \quad\left[L_{1}, L_{3}\right]=-\mathrm{i} \hbar L_{2} \quad \text { and cyclical permutations. } \tag{8.7}
\end{equation*}
$$

If we repeat the same argument discussed in the case of translation, we should see that the unitary operators generated by the exponential of the angular momentum represent the rotations, that is for a rotation of an angle $\theta$ around the $z$-axis we should have

$$
\begin{equation*}
U_{R(\theta)} \mathrm{e}^{-\frac{\left(\vec{r}-\vec{r}_{0}\right)^{2}}{a^{2}}}=\exp \left(-\frac{\mathrm{i}}{\hbar} \theta L_{z}\right) \mathrm{e}^{-\frac{\left(\vec{r}-\vec{r}_{0}\right)^{2}}{a^{2}}}=\mathrm{e}^{-\frac{\left(\vec{r}-\vec{r}_{1}\right)^{2}}{a^{2}}}, \tag{8.8}
\end{equation*}
$$

where I neglected the overall normalization of the wavefunction that is irrelevant in this computation. The position of the particle before the rotation is $\vec{r}_{0}$, while after the rotation
is $\vec{r}_{1}$. The coordinates of these two points are related as explained in the example below. It is straightforward to check that (8.8) holds in the case of very small angles $\theta$ : again, as in the computation done before for the translations, it is sufficient to Taylor-expand all $\theta$-dependent quantitites up to the first order and, in this case, use the definition of $L_{z}$, see Eq. (8.6). Notice that the wavefunctions that depend only $r^{2}$ (such as the one above with $\vec{r}_{0}=0$ ) are invariant under rotations, as expected! In order to check this it is sufficient to calculate the action of the $L_{i}$ 's on a (wave)function that depends only on $r^{2}$ and see that is trivial (zero). For instance

$$
\begin{align*}
L_{z} \psi\left(r^{2}\right) & =-\mathrm{i} \hbar\left(x \frac{\partial}{\partial y}-y \frac{\partial}{\partial x}\right) \psi\left(r^{2}\right)  \tag{8.9}\\
& =-\mathrm{i} \hbar \frac{d \psi\left(r^{2}\right)}{d r^{2}}\left(x \frac{\partial r^{2}}{\partial y}-y \frac{\partial r^{2}}{\partial x}\right)=-\mathrm{i} \hbar \frac{d \psi\left(r^{2}\right)}{d r^{2}}(2 x y-2 y x)=0 .
\end{align*}
$$

A similar computation holds also for the other components of the angular momentum $L_{x}$ and $L_{y}$.

Finally let us notice that also the set of all rotations forms a group. We will discuss the precise nature of this group later in this and the next sections.

### 8.2.1 Examples and exercises.

Consider for instance the operator

$$
O(\theta)=\left(\begin{array}{ccc}
\cos \theta & -\sin \theta & 0  \tag{8.10}\\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right)
$$

acting on the space $\mathbf{C}^{3}$. It represents a rotation of an angle $\theta$ around the $z$-axis. For instance, consider a point whose coordinates are $x_{0}=r \cos \alpha, y_{0}=r \sin \alpha$.


After the rotation the new coordinates are

$$
\begin{aligned}
x_{1} & =r \cos (\alpha+\theta)=r \cos \alpha \cos \theta-r \sin \alpha \sin \theta=x_{0} \cos \theta-y_{0} \sin \theta, \\
y_{1} & =r \sin (\alpha+\theta)=r \cos \alpha \sin \theta+r \sin \alpha \cos \theta=x_{0} \sin \theta+y_{0} \cos \theta .
\end{aligned}
$$

## Exercises.

- Consider the operator $L^{2}$

$$
\begin{equation*}
L^{2} \equiv L_{x}^{2}+L_{y}^{2}+L_{z}^{2}=\sum_{i=1}^{3} L_{i}^{2} . \tag{8.11}
\end{equation*}
$$

Show explicitly that it commutes with all the components of the angular momentum

$$
\begin{equation*}
\left[L^{2}, L_{x}\right]=0, \quad\left[L^{2}, L_{y}\right]=0, \quad\left[L^{2}, L_{z}\right]=0 . \tag{8.12}
\end{equation*}
$$

- Check explicitly that

$$
\begin{equation*}
\left(L_{x}+\mathrm{i} L_{y}\right)\left(L_{x}-\mathrm{i} L_{y}\right)=L^{2}-L_{z}^{2}+\hbar L_{z}, \quad\left(L_{x}-\mathrm{i} L_{y}\right)\left(L_{x}+\mathrm{i} L_{y}\right)=L^{2}-L_{z}^{2}-\hbar L_{z} . \tag{8.13}
\end{equation*}
$$

### 8.3 The angular momentum.

It is clearly important to find the basis of the angular momentum eigenvectors: this can be useful when we want to perform explictly a rotation (in this basis the operator in the exponent becomes just a number) or when the Hamiltonian commutes with $L_{i}$ (as in the problem of the Hydrogen atom). Of course we can not find simultaneous eignvectors for all components of the angular momentum, since they do not commute, see (8.6). The best we can achieve is to choos one component (for instance $L_{z}$ ) and look for the simultaneous eigenvectors of $L^{2}$ and $L_{z}$. This is possible thanks to Eq. (8.12). We already know one set of eigenvectors: from (8.9) it follows that any wavefunction $\psi\left(r^{2}\right)$ is an eigenvector with eigenvalue zero for both $L^{2}$ and $L_{z}$. We can attack the general problem by following an approach similar to the one used to find the energy eigenvectors of the harmonic oscillator hamiltonian.

In this derivation we will be using only two facts:

- the Hermitean properties $L_{i}^{\dagger}=L_{i}$,
- the commutation relations between $L_{i}$, given in (8.7), and their consquences.

Thus the results we will derive do not depend on the explicit form of the angular momentum operators (8.6) and hold for any triplet of Hermitean operators satisfying (8.7). In order to stress that the results are general we will use $J_{x}, J_{y}$ and $J_{z}$ to indicate three generic Hermitean operators satisfying (8.6).

Step 1. Suppose that we have an eigenvector of $J^{2}$ and $J_{z}$

$$
\begin{equation*}
J^{2}|j, m\rangle=\hbar^{2} j(j+1)|j, m\rangle, \quad J_{z}|j, m\rangle=\hbar m|j, m\rangle, \tag{8.14}
\end{equation*}
$$

where for later convenience we denoted the eigenvalue of $J^{2}$ with $\hbar^{2} j(j+1)$. Let us show that the possible eigenvalues of $J^{2}$ are non-negative (so that we can write them as the $\hbar \sqrt{j(j+1)}$ with $j \geq 0)$. This is easily done:

$$
\begin{equation*}
\hbar^{2} j(j+1)=\langle j, m| J^{2}|j, m\rangle=\| J|j, m\rangle \|^{2} \geq 0 . \tag{8.15}
\end{equation*}
$$

Step 2. Let us introduce the operators $L_{ \pm}$:

$$
\begin{equation*}
J_{+}=J_{x}+\mathrm{i} J_{y}, \quad J_{-}=J_{x}-\mathrm{i} J_{y} . \tag{8.16}
\end{equation*}
$$

It is straightforward to check that

$$
\begin{equation*}
\left[J_{z}, J_{+}\right]=\hbar J_{+}, \quad\left[J_{z}, J_{-}\right]=-\hbar J_{-} \tag{8.17}
\end{equation*}
$$

while from (8.12) it is clear that also $J_{ \pm}$commute with $J^{2}$. Now, starting from $|j, m\rangle$, we can generate new eigenvectors by acting with $J_{ \pm}$. By using (8.17) we have

$$
\begin{equation*}
J^{2} J_{ \pm}|j, m\rangle=\hbar^{2} j(j+1) J_{ \pm}|j, m\rangle, \quad J_{z} J_{ \pm}|j, m\rangle=\hbar(m \pm 1) J_{ \pm}|j, m\rangle \tag{8.18}
\end{equation*}
$$

So the state $J_{ \pm}|j, m\rangle$ is an eigenvector of $J^{2}$ with the same eigenvalue as $|j, m\rangle$ and is also an eigenvector of $J_{z}$ with eigenvalue $\hbar(m \pm 1)$. This proves that we can increase or dercrease the quantum number $m$ by an integer.

Step 3. The value of $m$ must be bigger than $-j$ and smaller than $j$ :

$$
\begin{equation*}
-j \leq m \leq j \tag{8.19}
\end{equation*}
$$

This is done by using again that the scalar product is non-degenerate together with (8.13)

$$
\begin{equation*}
\| J_{+}|j, m\rangle \|^{2}=\langle j, m| J_{-} J_{+}|j, m\rangle=\langle j, m|\left(J^{2}-J_{z}^{2}-\hbar J_{z}\right)|j, m\rangle=\hbar^{2}(j(j+1)-m(m+1)) . \tag{8.20}
\end{equation*}
$$

Since $\| J_{+}|j, m\rangle \|^{2} \geq 0$ then we must have

$$
\begin{equation*}
j(j+1)-m(m+1)=(j-m)(j+m+1) \geq 0, \tag{8.21}
\end{equation*}
$$

which imples $-j-1 \leq m \leq j$. In the same fashion we can calculate the norm square of $J_{-}|j, m\rangle$

$$
\begin{equation*}
\| J_{-}|j, m\rangle \|^{2}=\langle j, m| J_{-} J_{+}|j, m\rangle=\langle j, m|\left(J^{2}-J_{z}^{2}+\hbar J_{z}\right)|j, m\rangle=\hbar^{2}(j(j+1)-m(m-1)) . \tag{8.22}
\end{equation*}
$$

and we find that it is non-negative only if

$$
\begin{equation*}
j(j+1)-m(m-1)=(j-m+1)(j+m) \geq 0, \tag{8.23}
\end{equation*}
$$

which imples $-j \leq m \leq j+1$. By combining these two results we find (8.19).
Step 4. The quantum number $j$ must be either integer of half-integer. From step 2 above, we know that $J_{ \pm}$act as raising/lowering operators for the $J_{z}$ quantum number. If we begin with an eigenvector $|j, m\rangle$ we can apply $J_{+}$(or $J_{-}$) in order to increase (or decrease) the value of $m$. On the other hand we cannot violate the bound found above (8.19), thus at a certain point we must find

$$
\begin{equation*}
J_{+}|j, m\rangle=0 \quad \text { and } \quad J_{-}|j,-m\rangle=0 . \tag{8.24}
\end{equation*}
$$

This is possible only if $m=j$ (see (8.20)). Now I can start from the ket $|j, j\rangle$ and apply $J_{-} k$ times to lower the value of the $J_{z}$ eigenvalue to $m=j-k$. On the other hand we know that $m \geq-j$, which implies that after $k=2 j$ lowering operators have been applied to $|j, j\rangle$ we obtain a vector proportional to $|j,-j\rangle$ and a further $J_{-}$would simply lead to the zero vector. Since $k$ is integer (it counts the number of $J_{-}$), then $j$ must be integer of half-integer, as claimed above.

As a final remark, let us specialize this analysis to the angular momentum: in this case (8.6) implies that only integer values of $j$ are possible (see the discussion below in the Example section).

### 8.3.1 Examples and exercises.

Let us focus on the case of the angular momentum and derive some explicit expression for the eigenstates. We already know that any wavefunction $\psi\left(r^{2}\right)$ is an eigenstate with $j=m=0$. Thus if we find other eigenfunctions (with non-zero eigenvalues) we are free to multiple them by any function of $r^{2}$ only without changing the eigenvalues. You already saw this pattern in the study of the Hydrogen atom, where the energy eigenfunctions are the product of a radial function times a purely angular function that is an eigenvector of $L^{2}$ and $L_{z}$ (the spherical harmonics for the 2-sphere). We can derive the explicit form of the spherical harmonics by using the results of this section. We start by checking that the function

$$
\begin{equation*}
Y_{j}^{m=j}=\mathcal{N}_{j}^{j}\left(\frac{x+\mathrm{i} y}{r}\right)^{j}=\mathcal{N}_{j}^{j}(\sin \theta)^{j} \mathrm{e}^{2 \pi \mathrm{i} j \phi} \tag{8.25}
\end{equation*}
$$

is an eigenfunction of $L^{2}$ and $L_{z}$ with eigenvalues $j$ and $m=j$. The last relation on thee right hand side is just the rewriting of the $Y_{j}^{j}$ in polar coordinates. The quantum number $j$ here can take only integer values, otherwise the function $Y_{j}^{m=j}$ is not single valued (as it is clear if we look at the form written in polar coordinates and recall that $\phi=0$ and $\phi=2 \pi$ represent the same point). Let us look at the first non-trivial case $j=1$. By using the $L_{-}$operator we can find the other spherical harmonics

$$
\begin{equation*}
Y_{1}^{m=0}=\frac{L_{-}}{\hbar \sqrt{2}}\left(\mathcal{N}_{1}^{1} Y_{1}^{1}\right)=-\mathcal{N}_{1}^{1} \frac{\sqrt{2} z}{r} . \tag{8.26}
\end{equation*}
$$

The numerical factor of $\sqrt{2} \hbar$ ensures that the new spherical harmonics is normalized to one if the old one is normalized to one. It follows from (8.22) which in general requires to devide by $\hbar \sqrt{(j-m+1)(j+m)}$ everytime $L_{-}$acts on $|j, m\rangle$ if we want to work with vector of norm one.

$$
\begin{equation*}
Y_{1}^{m=-1}=\frac{L_{-}}{\hbar \sqrt{2}}\left(-\mathcal{N}_{1}^{1} \frac{\sqrt{2} z}{r}\right)=-\mathcal{N}_{1}^{1} \frac{x-\mathrm{i} y}{r} . \tag{8.27}
\end{equation*}
$$

You can rewrite these last two equation in polar coordinates and find the expression for the spherical harmonics that you did see in the analysis of the Hydrogen atom. You can repeat the same steps starting from (8.25) with $j=2$ and find the explicit form of the 5 harmonics of the level 2 .

## Exercises.

- Calculate $\mathcal{N}_{1}^{1}$ in (8.25) by requiring that $\int \bar{Y}_{1}^{1} Y_{1}^{1} d \Omega=1$
* Consider the wavefunction $\psi=x^{2} / r^{2}$. What is the probability of finding the values $j$ and $m$ in a simultaneous measurement of $L^{2}$ and $L_{z}$ ?


## 9 The spin.

We saw that a triplet of operators satisfying the commutation relations (8.7) can admit eigenvectors with a half-integer quantum number $j$. We also know that this kind of eigenvectors do not appear when we focus on the case of the angular momentum, where the generator take the particular form of Eq (8.6). One might wonder whether the case of half-integer $j$ appears in some interesting physical system or not. The surprising answer is that this case is indeed very common!

## 9.1 $S O(3)$ representations.

Consider the three wavefunctions (8.25), (8.26) and (8.27) derived in the previous example. They form a basis for the subspace of wavefunctions with $j=1$ (so the $J^{2}$ eigenvalue is $2 \hbar^{2}$ for all these states). We can represent these three states as follows

$$
Y_{1}^{1} \leftrightarrow|1,1\rangle=\left(\begin{array}{l}
1  \tag{9.28}\\
0 \\
0
\end{array}\right) \quad, \quad Y_{1}^{0} \leftrightarrow|1,0\rangle=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right) \quad, \quad Y_{1}^{-1} \leftrightarrow|1,-1\rangle=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right) .
$$

From the results summarized in the previous example, we know how $L_{z}$ and $L \pm$ act that in this subspace

$$
\begin{align*}
& L_{z}=\hbar(|1,1\rangle\langle 1,1|-|1,-1\rangle\langle 1,-1|)  \tag{9.29}\\
& L_{+}=\sqrt{2} \hbar(|1,1\rangle\langle 1,0|+|1,0\rangle\langle 1,-1|) \\
& L_{-}=\sqrt{2} \hbar(|1,0\rangle\langle 1,1|+|1,-1\rangle\langle 1,0|)
\end{align*}
$$

These operators cen ba written in terms of matrices acting on the vector (9.28)

$$
L_{z}=\hbar\left(\begin{array}{ccc}
1 & 0 & 0  \tag{9.30}\\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right), \quad L_{+}=\hbar\left(\begin{array}{ccc}
0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2} \\
0 & 0 & 0
\end{array}\right), \quad L_{-}=\hbar\left(\begin{array}{ccc}
0 & 0 & 0 \\
\sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & 0
\end{array}\right)
$$

The three states in (9.28) form the so-called vector representation of the $S O(3)$ rotation group. Let us check that there is a direct relation between the matrices (9.30) and the generators of the rotations as seen in (8.10). In order to see this let us introduce the matrix

$$
A=\left(\begin{array}{ccc}
\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\
\frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\
0 & 1 & 0
\end{array}\right) .
$$

From eqs. (8.25), (8.26) and (8.27), we can see that $A$ implements a change of basis from the eigenvectors $Y_{1}^{1}$ to the standard cartesian basis where the first eigenfunction is proportional to $x$ and the remaining ones to $y$ and $z$. In this basis the generators of the rotation around $z\left(L_{z}\right)$ takes a different form with the respect of (9.30)

$$
L_{z}=A \hbar\left(\begin{array}{ccc}
1 & 0 & 0  \tag{9.31}\\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right) A^{-1}=\hbar\left(\begin{array}{ccc}
0 & \mathrm{i} & 0 \\
-\mathrm{i} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Similar relations hold for $L_{x}$ and $L_{y}$ showing that, in this basis, they form the standard generators for the $S O(3)$ rotation group (see the example 8.2.1).

## Exercises.

[Op.] One can follow a similar derivation also for the 5 eigenfunctions with $j=2$. Show that there is a one-to-one correspondence between these eigenfunctions and the symmetric square matrices.

### 9.2 Spin $1 / 2$ and $S U(2)$ representations.

So far we have described particles through their position and momentum. Mathematically these observables are related to two Hermitean operators $\hat{x}$ and $\hat{p}$ satisfying the canonical commutation relations (5.4). Since, by hypothesis we are dealing with point-like object, apparently there is no room for any other basic observable and one might think that all other observables should be build by using $\hat{x}$ and $\hat{p}$. (For instance, the angular momentum is given by (8.6), the Hamiltonian is give, in the free case, by Eq. (5.3)). However this is not what happens in nature. On the contrary all known "matter" particles (such as the electron, the muon, the quarks, the neutrinos) are not completly determined by specifying their position ${ }^{12}$. It turns out that matter particles possess additional degrees of freedom called "spin". To be precise this means, for this particles we have the following properties:

- The spin degrees of freedom are described by a triplet of operators $S_{x}, S_{y}$ and $S_{z}$ satisfying the relations (8.7). A complete set of commuting operators (CSCO) is given, for instance, by $\hat{x}, S_{z}$ and $S^{2}$.
- The Hilbert space describing the state of the particle is the tensor product of the Hilbert space $\mathcal{H}_{x}$ where $\hat{x}$ and $\hat{p}$ act and the Hilbert space $\mathcal{H}_{S}$ where the $S_{i}$ act.
- Elementary matter particles with half-integer spin behave as fermions, while those with integer spin behave as boson (and, as we have seen in the previous section this affects the description of multiparticle systems!). For instance, for a spin $1 / 2$ particle the only (eigen)value of $S^{2}$ is $3 \hbar^{2} / 4$ corresponding to an eigenvalues $s=1 / 2$. This implies that $\mathcal{H}_{S}$ is two dimensional and a basis for this space is given by

$$
\left|s=\frac{1}{2}, m=\frac{1}{2}\right\rangle, \quad \text { and } \quad\left|s=\frac{1}{2}, m=-\frac{1}{2}\right\rangle,
$$

where as in the previous section, the first and the second number indicate the eigenvalues of $S^{2}$ and $S_{z}$ respectively.

### 9.2.1 Examples and exercises.

An explicit realization of $\mathcal{H}_{S}$ is $\mathbf{C}^{2}$. As usual in the case of finite dimensional spaces, we can realize any operator as a matrix. Conventionally the following choice is taken for the spin operators: $S_{a}=\frac{\hbar}{2} \sigma_{a}$, with $a=1,2,3$ and

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1  \tag{9.32}\\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

[^1]These $\sigma$ 's are called Pauli matrices. In this representation we have

$$
\begin{equation*}
\left|\frac{1}{2}, \frac{1}{2}\right\rangle \leftrightarrow\binom{1}{0} \quad, \quad \text { and } \quad\left|\frac{1}{2},-\frac{1}{2}\right\rangle \leftrightarrow\binom{0}{1} . \tag{9.33}
\end{equation*}
$$

These two states are commonly indicated as "spin up" and "spin down" states.

## Exercises.

- Check the following property of Pauli matrices

$$
\begin{equation*}
\left[\sigma_{a}, \sigma_{b}\right]=2 \mathrm{i} \epsilon_{a b c} \sigma_{c}, \quad \sigma_{a} \sigma_{b}+\sigma_{b} \sigma_{a}=2 \delta_{a b} \tag{9.34}
\end{equation*}
$$

- Show that

$$
\mathrm{e}^{\mathrm{i} \phi \sigma_{2}}=\left(\begin{array}{cc}
\cos \phi & \sin \phi  \tag{9.35}\\
-\sin \phi & \cos \phi
\end{array}\right)
$$

### 9.3 Addition of two spins.

Consider now a two particle systems whose constituents have spin $s^{(1)}$ and $s^{(2)}$ respectively. As we know, the Hilbert space describing the whole system is the tensor product of the spaces describing the single constituents. In particular, we also need to consider the tensor product of the spaces describing the spin: $\mathcal{H}_{S_{1}}$ for the first particle and $\mathcal{H}_{S_{2}}$ for the second one. The total spin of the system is of course $\vec{S}=\vec{S}_{1}+\vec{S}_{2}$, where $\vec{S}_{1}$ acts only on the first space $\mathcal{H}_{1}$ and $\vec{S}_{2}$ on the second one. Then also the components of $\vec{S}$ satisfy the commutation relations (8.6) and so we we should be able to write the states of the total system in terms of the eigenvectors derived in the previous section. The question we want to address now is what are the eigenvalue for the total spin $S^{2}$, if we know the eigenvalue of each of the constituents. We have the following result:

If $s_{1}$ and $s_{2}$ indicate the spin quantum number of the constituents (that is $S_{1}^{2}$ eigenvalue is $\hbar^{2} s_{1}\left(s_{1}+1\right)$ and similarly for $\left.S_{2}^{2}\right)$, then the possible eigenvalues for the total spin $S^{2}$ are $s$ with $\left|s_{1}-s_{2}\right| \leq s \leq s_{1}+s_{2}$ and each eigenvalue appears just as one multiplet (a set of $2 s+1$ values of $S_{z}$ ).

Let us work out explicitly the simple case of two objects having each one spin $1 / 2$. For instance consider an hydrogen atom: both the proton and the electron have spin $1 / 2$ and we would like to know the spin of the whole atom. The spin states

$$
\begin{align*}
& |\uparrow, \uparrow\rangle=\binom{1}{0} \otimes\binom{1}{0} \quad, \quad|\uparrow, \downarrow\rangle=\binom{1}{0} \otimes\binom{0}{1},  \tag{9.36}\\
& |\downarrow, \uparrow\rangle=\binom{0}{1} \otimes\binom{1}{0} \quad, \quad|\downarrow, \downarrow\rangle=\binom{0}{1} \otimes\binom{0}{1},
\end{align*}
$$

where the first (second) entry refers to the the first (second) particle and we used $\uparrow$ in order to indicate a state with the positive eigenvalue of $S_{z}$. Clearly the z-component of the total spin of the state $|\uparrow, \uparrow\rangle$ is $\hbar$ and thus it must represent the eigenvector $|1,1\rangle$. From this vector we can generate the other two states of the triplet with $s=1$

$$
\begin{equation*}
|1,0\rangle=\frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle+|\downarrow, \uparrow\rangle), \quad|1,-1\rangle=|\downarrow, \downarrow\rangle \tag{9.37}
\end{equation*}
$$

Finally the state $|0,0\rangle$ must be the state with one spin up and one spin down that is orthogonal to $|1,0\rangle$ (since they have a different eigenvalue).

$$
\begin{equation*}
|0,0\rangle=\frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle-|\downarrow, \uparrow\rangle) . \tag{9.38}
\end{equation*}
$$

This steps can be repeated in the case of general spins and yield to the result summarized above.

### 9.3.1 Examples and exercises.

## Exercises.

- Consider a composite system with two particle of $\operatorname{spin} s_{1}$ and $s_{2}$ respectively. What is the dimension of the Hilbert space describing the spin degrees of freedom?
- Both the proton and the electron have spin $1 / 2$. Is the hydrogen atom a boson or a fermion?


## Legenda:

| $\equiv$ | "equivalent by definition" | $\mathbf{C}$ |
| :--- | :--- | :--- |
| $\exists$ | the set of complex numbers |  |
| $\exists!$ | "exists at least one" | $\mathbf{N}$ | the set of positive integer numbers


[^0]:    ${ }^{11}$ Starting from a vector $|\psi\rangle$ of norm 1 , we can check that the norm of $U_{a}|\psi\rangle$ is independet of $a$ and thus is one.

[^1]:    ${ }^{12}$ Of course in quantum mechanics specifying the position of the particle means that give a wavefunction $\psi(x)=\langle x \mid \psi\rangle$.

