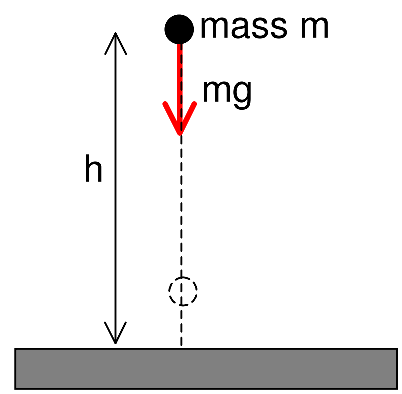
ELECTRIC ENERGY AND ELECTRIC POTENTIAL
(*Young & Freedman* Chap. 23)
(*Ohanian* Chaps. 25, 26)

Review of Force, Work, and Potential Energy

The electric field exerts a force on a charged object. If the charge moves, WORK is done

Recall: Work done = Potential energy gained = (Force)(Distance)



Example:

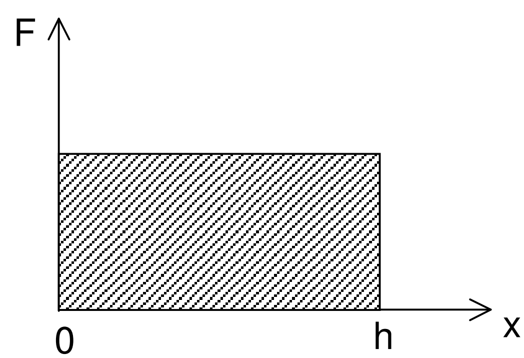
Mass m lifted through height h against gravity

Required force $F = \frac{GmM_E}{R_E^{\,2}} = mg$

where M_E = Mass of the Earth R_E = Radius of the Earth

Work done = PE gained = (Force)(Distance)

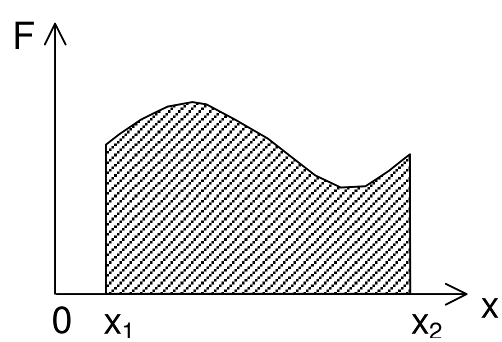
$W = \Delta U = mgh$



W = Area under Force-Distance graph

$$W = \Delta U = \int_0^h F(x)dx$$

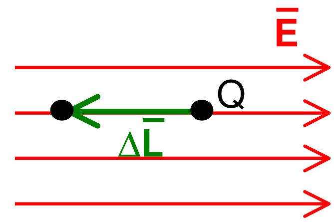
In general, for ANY force:



$$W = \Delta U = \int_{x_1}^{x_2} F(x)dx$$

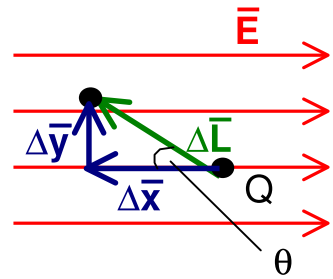
Electric potential energy

Let a charge Q be moved through a displacement $\Delta\overline{\mathbf{L}}$ AGAINST an electric field $\overline{\mathbf{E}}$.



Work done = PE gained = (Force)(Distance)
W = ΔU = $(QE)\Delta L$

What if the displacement is not exactly against $\overline{\mathbf{E}}$?



Let $\Delta\overline{\mathbf{L}}$ be the sum of

$\Delta\overline{\mathbf{x}}$ along $\overline{\mathbf{E}}$ and
$\Delta\overline{\mathbf{y}}$ perpendicular to $\overline{\mathbf{E}}$.

For $\Delta\overline{\mathbf{y}}$: no force as $\overline{\mathbf{E}}$ has no component along this direction.

For $\Delta\overline{\mathbf{x}}$: $\overline{\mathbf{E}}$ and $\Delta\mathbf{x}$ are antiparallel $\Rightarrow$ $\Delta U = QE\Delta x = QE(\Delta L\cos\theta)$

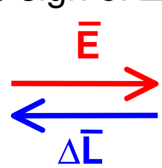
Now $\overline{\mathbf{E}}\cdot\Delta\overline{\mathbf{L}} = -E\Delta L\cos\theta$ (negative sign as $\overline{\mathbf{E}}$ and $\Delta\overline{\mathbf{L}}$ are anti-parallel)

So $\Delta U = -Q\overline{\mathbf{E}}\cdot\Delta\overline{\mathbf{L}}$

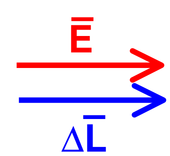
i.e., **The change in electric potential energy is equal to minus the charge multiplied by the dot product of the electric field and the displacement vectors**

Let's consider why the sign of ΔU has to be negative:

If Q moves against **E**



If Q moves with **E**



We must do work to push it
$\Rightarrow$ It gains PE
$\Rightarrow$ ΔU is positive

The field pulls it along
$\Rightarrow$ It loses PE
$\Rightarrow$ ΔU is negative

$\overline{\mathbf{E}}\cdot\Delta\overline{\mathbf{L}}$ is positive if $\overline{\mathbf{E}}$ and $\Delta\overline{\mathbf{L}}$ are parallel
negative if $\overline{\mathbf{E}}$ and $\Delta\overline{\mathbf{L}}$ are anti-parallel

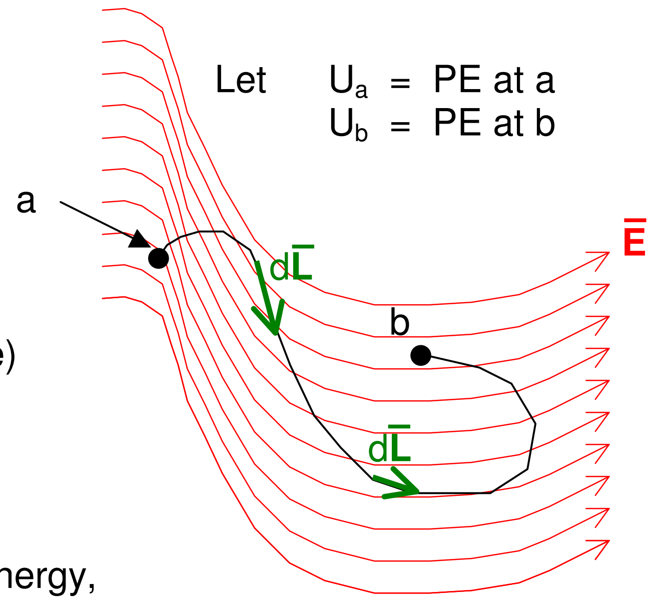
So: $\Delta U = -Q\overline{\mathbf{E}}\cdot\Delta\overline{\mathbf{L}}$, as derived above

More general case:

So far we have considered a displacement in a straight line in a uniform electric field. What if the field is non uniform and the path of the charge is arbitrary?

Let Q move from a to b along an arbitrary path in a non-uniform electric field $\overline{\mathbf{E}}(x,y,z)$

Let U_a = PE at a
U_b = PE at b



Let the path be broken into many small displacements $d\overline{\mathbf{L}}$ (two typical ones shown here)

For any one element, we have
$dU = -Q\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$

To find the change in potential energy, ΔU, for the whole path, we must add up all of the contributions from each $d\overline{\mathbf{L}}$ by INTEGRATING this expression over the whole of the path:

$$\Delta U = U_b - U_a = -Q\int_a^b \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$$

Electric potential and potential difference

Definition: The ELECTRIC POTENTIAL, V, at a point in space is the potential energy which a unit positive charge would have at that point

i.e., Potential = Potential energy per unit charge $\quad V = \dfrac{U}{Q}$

Potential energy is a relative quantity. It is more sensible to work in terms of POTENTIAL DIFFERENCE:

Definition: The ELECTRIC POTENTIAL DIFFERENCE, ΔV, between two points a and b is equal to the work done in moving a unit positive charge from a to b.

From the expression derived above for the electric energy difference, ΔU, we have

$$\Delta V = V_b - V_a = -\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}$$

This very important expression relates the electric field vector to the electric potential, a scalar quantity related to energy. The potential difference between two points is the line integral of the electric field along any path between the points.

The work done in moving a charge Q through a potential difference ΔV is

$$W = Q\Delta V$$

Note:

1. V can be defined to be zero at any point we choose. As there are no restrictions, we usually pick a position to make calculations simplest (e.g., at V = 0 at infinity or at the surface of a conductor)

2. Units of V: $\Delta V = -\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}$

   $\Rightarrow$ Potential $\equiv$ (Electric field)(Distance) $\equiv \left[\dfrac{\text{Force}}{\text{Charge}}\right]\text{Distance}$

   Ao the units are $\text{N C}^{-1}\text{ m}$

   In the SI system: 1 Volt (V) = $1\text{ N C}^{-1}\text{ m}$

   i.e., if a force of 1 N moves a charge of 1 C through 1 m, then the potential difference between the two points is 1 V.

$$\text{Electric Field} \equiv \frac{\text{Potential}}{\text{Distance}} \equiv \frac{\text{Volts}}{\text{Metres}}$$

Therefore, Electric field can be expressed in units of V m^{-1}.
In fact, these are the units in which the electric field is usually quoted.

3. The ELECTRON VOLT (eV) as a unit of energy or work

If a charge of e moves through a potential difference of 1 V, then the work done is:

$$W = Q(\Delta V) = (e)(1) = 1.602 \times 10^{-19} \text{ Joules} = 1 \text{ eV}$$

How to find V
Method 1: Integrate the electric field

1. If $\overline{\mathbf{E}}$ is not given, then find it (e.g., use Gauss's Law).

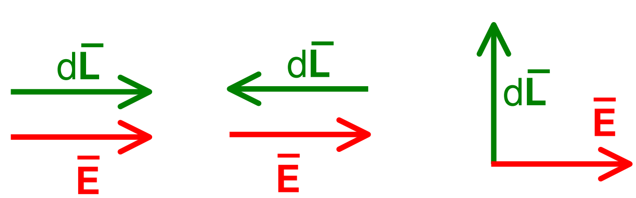
2. Choose the location of V = 0, if it is not given
   e.g.: V = 0 at infinity
   V = 0 at the surface of a conductor

3. Work out ΔV using $$\Delta V = -\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}$$

Forget about the sign: just work out the magnitude: $$|\Delta V| = \left| \int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} \right|$$

Choose a suitable path to make the line integral easy:

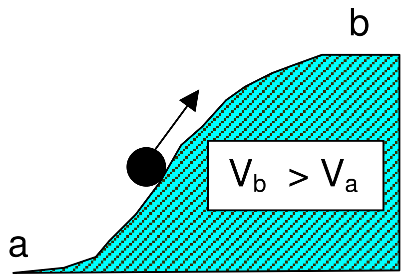
i.e., make $\overline{\mathbf{E}}$ and $d\overline{\mathbf{L}}$ parallel or antiparallel or perpendicular



4. Now use common sense to determine whether ΔV is positive or negative:

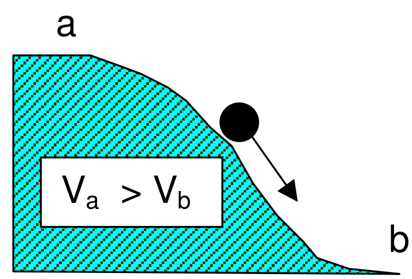
If you would need to PUSH positive charge AGAINST the field to go from a to b

Analogy: Pushing a ball up a hill



If the electric field would PULL positive charge along from a to b

Analogy: A ball rolling down a hill



Examples of finding the electric potential by integrating the electric field

1. Potential due to a point charge
2. Potential due to an infinite sheet of charge
3. Potential due to a sphere of uniform charge density

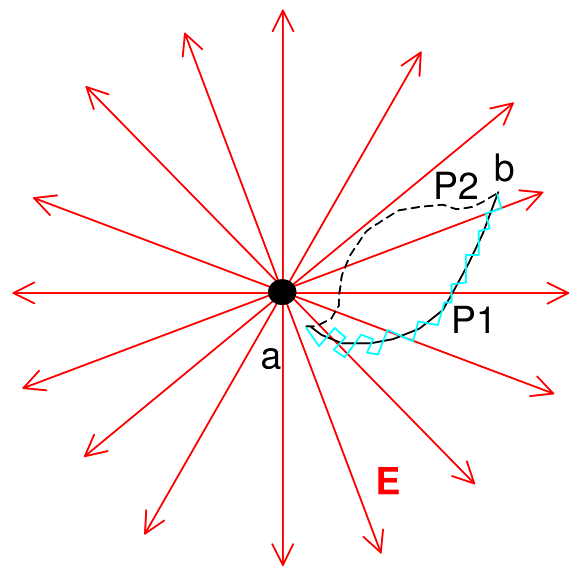
See lecture notes

The electric field is conservative

Definition: A field is conservative if the work done in moving between any two points is independent of the path taken.

$\overline{\mathbf{E}}$ is conservative because ΔV is independent of the path.

Proof: Consider the field due to a point charge Q. If we can prove it for this we can argue from the principle of superposition that it must be true for any charge distribution (i.e., any electric field distribution).



Let P1 and P2 be two arbitrary paths between a and b.

Divide each path into many RADIAL and CIRCULAR ARC sections.

For the radial sections:
$\Delta V = Edr$ as $\overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = Edr$ (parallel)

For the circular arc sections:
$\Delta V = 0$ as $\overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = 0$ (perpendicular)

Every dr for path A has an exactly corresponding dr for path B, and total change in radius is exactly the same for the two paths

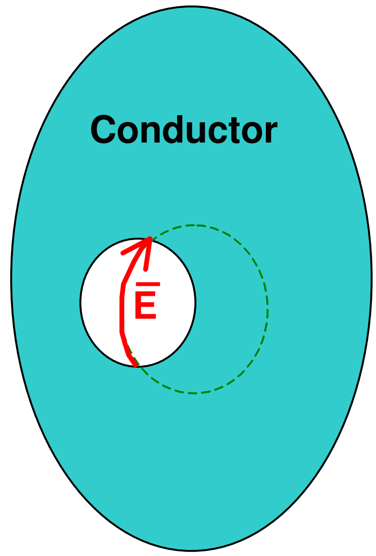
i.e.: $$\left[\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}\right]_{\text{PATH P1}} = \left[\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}\right]_{\text{PATH P2}} \Rightarrow \overline{\mathbf{E}} \text{ is conservative}$$

Another way of putting it: $$\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} + \int_b^a \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = 0 \Rightarrow \oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = 0$$

The line integral of the electric field around a closed path is zero.

The electric field is zero inside a closed empty cavity inside a conductor

Proof: We will assume that $\overline{\mathbf{E}} \neq 0$ in the cavity and so deduce a result which we know is incorrect. This will imply that our original assumption must be wrong, so that $\overline{\mathbf{E}} = 0$.



So, assume $\overline{\mathbf{E}} \neq 0$ in the cavity.

$\overline{\mathbf{E}} = 0$ in the conductor (proved before)

There are no charges in the cavity, so the field lines in the cavity must terminate on charges on the inner surface

Consider a closed loop as shown by the dotted line:

$\overline{\mathbf{E}} = 0$ in the conductor $\qquad \overline{\mathbf{E}} \neq 0$ in the cavity

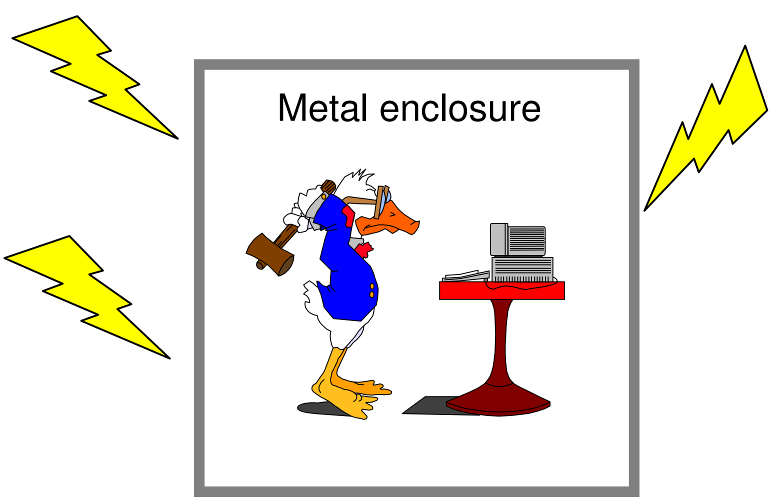
Therefore $\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} \neq 0$ for the closed loop

$\Rightarrow$ violation of the conservative nature of $\overline{\mathbf{E}}$, which we know to be impossible.

Therefore we must conclude that $\overline{\mathbf{E}} = 0$ inside the cavity.

This is the principle of the **FARADAY CAGE**:

Anything or anyone inside is completely isolated from electric field disturbances outside the metal enclosure.



How to find V
Method 2: Use the principle of superposition

This method can be used to find the potential at a point in spacesome given distribution of charge. We assume that V = 0 at infinite distance.

1. Break the charge distribution into many small elements, dQ.
2. Regard each dQ as a point charge
3. Use the equation for the potential due to a point charge to write down the contribution of dQ to the potential at P:

$$dV = \frac{dQ}{4\pi\varepsilon_o r}$$

4. Apply the principle of superposition: the total potential is the sum of all the contributions from all the dQ elements:

$$V = \int_{\text{Charge disribution}} \frac{dQ}{4\pi\varepsilon_o r}$$

Example: finding the electric potential using the principle of superposition

1. Potential on the axis of a line of charge

See lecture notes

The electric field is the gradient of the electric potential

Recall: In general, a vector field can be expressed as the gradient of a scalar field:

If Hx,y,z) is a scalar field, then

$$\text{Grad(H)} = \overline{\nabla}\text{H} = \frac{\partial H}{\partial x}\hat{\mathbf{i}} + \frac{\partial H}{\partial y}\hat{\mathbf{j}} + \frac{\partial H}{\partial z}\hat{k} = \left[\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]H$$

Now, $dV = -\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} \quad \Rightarrow \quad \overline{\mathbf{E}} = -\frac{dV}{d\overline{\mathbf{L}}}$

If we know V(x,y,z) we can find $\overline{\mathbf{E}}$ (x,y,z) by taking partial derivatives.

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$$\overline{\mathbf{E}}(x,y,z) = -\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]V = -\overline{\nabla}V$$

Examples: finding the electric field when given V(x,y,z)

See lecture notes

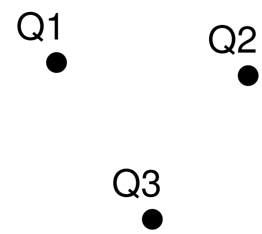
Electric energy of a system of point charges

Note: The treatments of this topic in *Y&F* and *Ohanian* are a bit different. The following is based on *Ohanian's* approach.

The total energy, U_{tot}, of a system of charges is equal to the total amount of work, W_{tot}, that must be done to assemble it.

Recall: The work done in moving a charge Q through a potential difference V is

$$W = Q\Delta V.$$



Consider a system of three charges, Q_1, Q_2 and Q_3, which are brought together from $r = \infty$ where we let $V = 0$.

Assume that we assemble them in the order 1-2-3

Work done to bring in Q_1 = $W_1 = 0$ (there is no opposing field)
Work done to bring in Q_2 = $W_2 = Q_2V_{21}$
Work done to bring in Q_3 = $W_3 = Q_3V_{31} + Q_3V_{32}$

where V_{21} = potential at position 2 due to due to Q_1 being nearby.
V_{31} = potential at position 3 due to due to Q_1 being nearby.
etc.

So, $W_{tot} = \underline{U_{tot}} = Q_2V_{21} + Q_3V_{31} + Q_3V_{32}$

Now, imagine that we brought them together in the reverse order, 3-2-1:

$$W_{tot} = U_{tot} = 0 + Q_2V_{23} + (Q_1V_{12} + Q_1V_{13})$$

No work needed to bring in Q_3

Q_2: we do work against the repulsion of Q_3

Q_1: we do work against the repulsion of Q_3 and

If we add these two equations for W_{tot}, we get:

$$2W_{tot} = 2U_{tot} = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$$

Now $V_{12} + V_{13} = V_1$ = Total potential at Q_1 due to Q_2 and Q_3 combined
$V_{21} + V_{23} = V_2$ = Total potential at Q_2 due to Q_1 and Q_3 combined
$V_{31} + V_{32} = V_3$ = Total potential at Q_3 due to Q_1 and Q_2 combined

$$\Rightarrow \quad 2W_{tot} = 2U_{tot} = Q_1V_1 + Q_2V_2 + Q_3V_3$$

This treatment can easily be extended to an arbitrary number of charges:

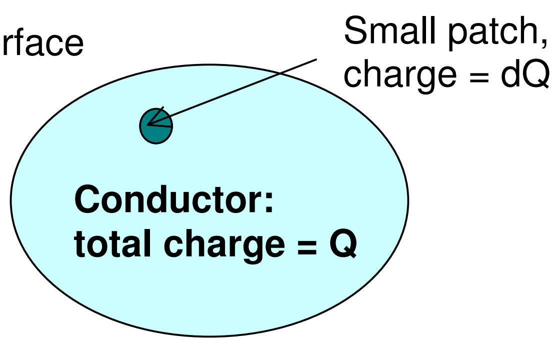
For n charges, $$U_{tot} = \frac{1}{2}\sum_{1}^{n} Q_i V_i$$

Energy of a system of point charges

where V_i = potential at the position of Q_i due to all the other charges.

Electric energy of a charged conductor

Recall: All of the excess charge is on the surface



Since it is a conductor, the surface is an equipotential $\Rightarrow$ V is the same everywhere

Energy of dQ is $dU = ½(dQ)V$

$$\Rightarrow \quad U_{tot} = \frac{1}{2}\oint V dQ = \frac{1}{2}V\oint dQ = \frac{1}{2}QV$$

For a system of n conductors, $$U_{tot} = \frac{1}{2}\sum_{1}^{n} Q_i V_i$$

(just as for a system of n point charges)

Examples: finding the electric energy of a system

1. Electric energy of a conducting sphere
2. Electric energy of a parallel plate capacitor
3. Electric energy of a uniformly charged sphere

See lecture notes

Energy density of the electric field

Recall: For a parallel plate capacitor, the total stored energy is

$$U_{tot} = \frac{1}{2}\frac{Q^2 d}{\varepsilon_o A}$$

We can rewrite this as $$U_{tot} = \frac{1}{2}\varepsilon_o\left[\frac{Q}{\varepsilon_o A}\right]^2(Ad)$$

$$\Rightarrow \; U_{tot} = \frac{1}{2}\varepsilon_o E^2\,(\text{volume of space between the plates})$$

where E is the (uniform) electric field between the plates. Here we regard the stored energy as residing in the space between the plates.

Energy density (energy per unit volume): $u = \frac{1}{2}\varepsilon_o E^2$

(the energy density has units of J m^{-3} in the SI system).

It can be shown that this result holds generally, for ANY electric field

The ENERGY DENSITY OF THE ELECTRIC FIELD is $u = \frac{1}{2}\varepsilon_o E^2$