REVIEW OF VECTOR ALGEBRA
(*Young & Freedman* Chapter 1)

Scalars and vectors

SCALAR: Magnitude only
Examples: Mass, time, temperature, voltage, electric charge

VECTOR: Magnitude and Direction
Examples: Displacement, force, velocity, electric field, magnetic field

Vector notation

It is vital to distinguish vectors from scalars. Various conventions are used to denote vectors:

Boldface letters: e.g., **A**

Bars, arrows or squiggles: $\bar{A}$ $\vec{A}$ $\underset{\rightarrow}{A}$ $\tilde{A}$

Young & Freedman uses boldface with and arrow: $\vec{\mathbf{A}}$

Ohanian uses just **boldface** letters : **A**

- I will use $\bar{\mathbf{A}}$ (with the letter in boldface in printed notes)
- For the magnitude of a vector (which is a scalar), I will use the letter in plain typeface and without the bar:

e.g.: Magnitude of $\bar{\mathbf{A}}$ is A

Sometimes I will use the convention of putting the letter between vertical

bars: e.g.: Magnitude of $\bar{\mathbf{A}}$ is $|\bar{\mathbf{A}}|$

For unit vectors (of magnitude equal to one) I will use lower case letters with a "hat" on top:

e.g.: $\hat{\mathbf{a}}$

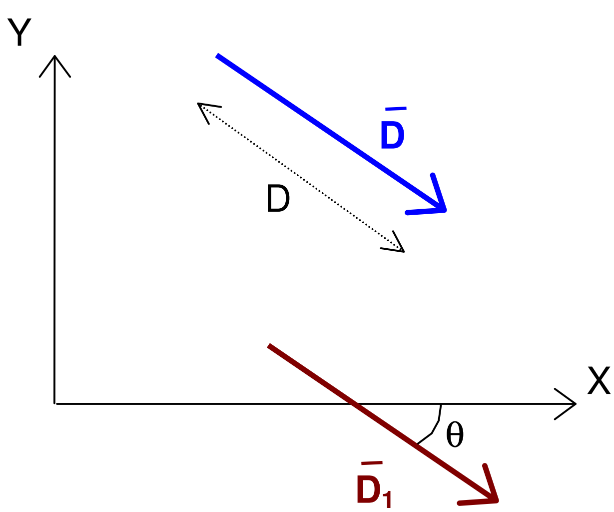
Recommendation: You should use the same conventions - this is not obligatory - if you prefer you may adopt one of the other conventions as long as you use it correctly and consistently.

Assuming that you follow this recommendation, then

Don't forget: if it is a vector, put a bar on it.
If it is a unit vector, put a "hat" on it.

Simple example of a vector: displacement vector in the X-Y plane

Vectors are drawn as arrowed lines with the arrow giving the direction and the length representing the magnitude



Magnitude of $\bar{\mathbf{D}}$ = length D

Direction of $\bar{\mathbf{D}}$ is specified by the angle θ

Vector equality

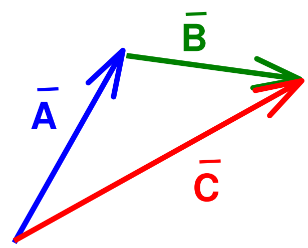
Two vectors are equal if and only if they are equal in magnitude and direction

e.g., vectors $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}_1$ in the diagram above are equal even though they are not coincident in space.

Vector addition

If we add two vectors we get another vector.

To add $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$



$\overline{\mathbf{C}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$

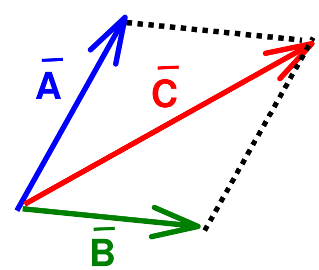
Put the beginning of one on the end of the other

The vector $\overline{\mathbf{C}}$ is formed by joining the beginning and end of the combination is the sum

$\overline{\mathbf{C}}$ is also called the RESULTANT of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$

The PARALLELOGRAM LAW is another way of adding $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$

1. Let the start of $\overline{\mathbf{A}}$ coincide with the start of $\overline{\mathbf{B}}$
2. Draw a parallelogram with $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ as sides
3. The resultant, $\overline{\mathbf{C}}$ is the diagonal containing the starts of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$

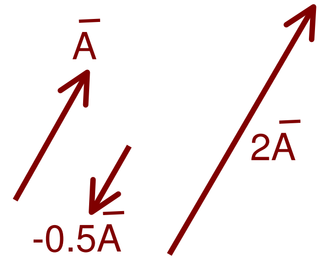


Note: Clearly, $\overline{\mathbf{A}} + \overline{\mathbf{B}} = \overline{\mathbf{B}} + \overline{\mathbf{A}}$ (vector addition is COMMUTATIVE)

$(\overline{\mathbf{A}} + \overline{\mathbf{B}}) + \overline{\mathbf{D}} = \overline{\mathbf{A}} + (\overline{\mathbf{B}} + \overline{\mathbf{D}})$ (vector addition is ASSOCIATIVE)

Vector multiplication by a real number

x$\bar{\mathbf{A}}$ has



Magnitude = xA (i.e., x times the magnitude of $\bar{A}$)

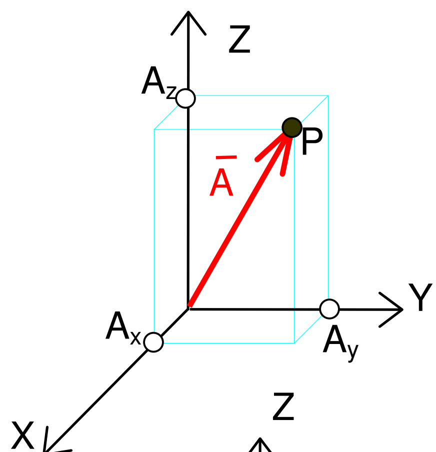
Direction = the same as that of $\bar{\mathbf{A}}$ if x is positive

= opposite to that of $\bar{\mathbf{A}}$ if x is negative

Components of a vector along the coordinate axes

We will consider the 3-dimensional case using as an example the position vector with respect to the origin. The result applies to ANY sort of vector.

Let point P have coordinates A_x, A_y, A_z in 3-dimensional space



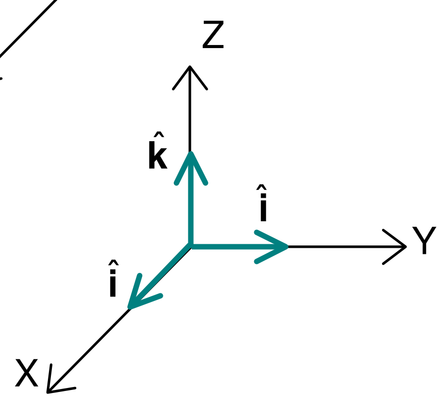
Let $\bar{A}$ be the displacement vector of point P from the origin.

We describe $\bar{A}$ in terms of three ORTHOGONAL UNIT VECTORS along the three axes:

$\hat{\mathbf{i}}$ has magnitude 1 and points along +X

$\hat{\mathbf{j}}$ has magnitude 1 and points along +Y

$\hat{\mathbf{k}}$ has magnitude 1 and points along +Z

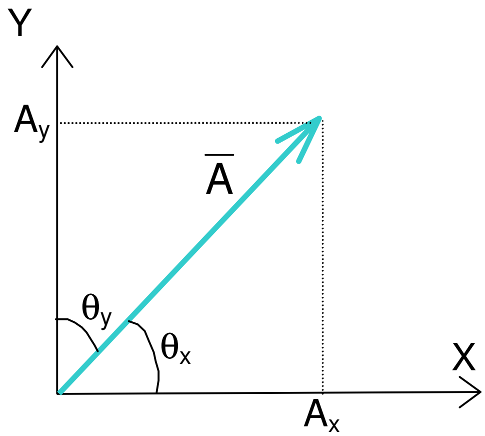


Clearly $\quad \bar{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$

Note: alternative notion used in some books: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \equiv \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

A_x, A_y, A_z are called the COMPONENTS of the vector $\bar{\mathbf{A}}$

Two-dimensional example:



$A_x = A\cos(\theta_x)$
$A_y = A\sin(\theta_x) = A\cos(\theta_y)$

By PYTHAGORAS'S THEOREM

$$A = \sqrt{A_x^2 + A_y^2}$$

i.e., the magnitude of a vector is equal to the square root of the sum of the squares of its components

In the 3-dimensional case:

$A_x = A\cos(\theta_x)$
$A_y = A\cos(\theta_y)$
$A_z = A\cos(\theta_z)$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Addition of vectors in terms of their components

Let $\overline{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ $\quad\quad \overline{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$

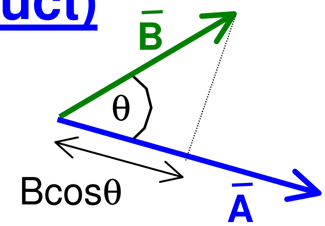
Then, because vector addition is commutative, we have

$$\overline{\mathbf{A}} + \overline{\mathbf{B}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$$

i.e., we simply add the components separately.

The dot product (scalar product)

Definition: $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = AB\cos\theta$



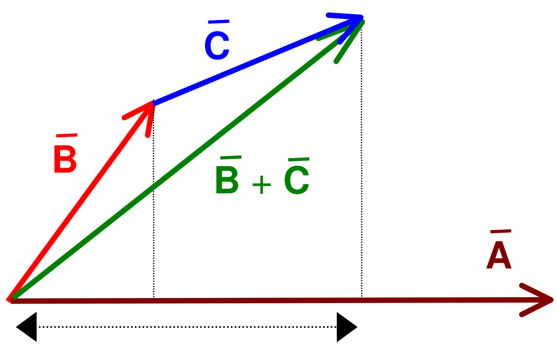
i.e. $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}}$ = (Magnitude of $\bar{\mathbf{A}}$)(Magnitude of projection of $\bar{\mathbf{B}}$ onto $\bar{\mathbf{A}}$)

Things to note about the dot product:

1. $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}}$ is a SCALAR

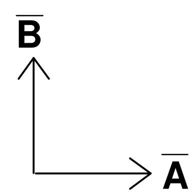
2. $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = \bar{\mathbf{B}} \cdot \bar{\mathbf{A}}$ (Commutative)

3. $\bar{\mathbf{A}} \cdot (\bar{\mathbf{B}} + \bar{\mathbf{C}}) = \bar{\mathbf{A}} \cdot \bar{\mathbf{B}} + \bar{\mathbf{A}} \cdot \bar{\mathbf{C}}$ (Distributive)

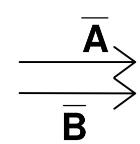


Projection of $\bar{\mathbf{B}} + \bar{\mathbf{C}}$ onto $\bar{\mathbf{A}}$
= sum of projections of $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$.

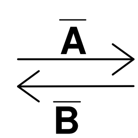
4. If $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are perpendicular $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = 0$ as $\cos(90^o) = 0$



5. If $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are parallel $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = AB$ as $\cos(0^o) = 1$



6. If $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are anti-parallel $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = -AB$ as $\cos(180^o) = -1$



7. $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1$

8. $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$ as they are parallel

$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$ as they are orthogonal

9. The dot product of two vectors is the sum of the products of their components:

$$\overline{\mathbf{A}} \cdot \overline{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) = A_xB_x + A_yB_y + A_zB_z$$

Exercise: Prove this using 3, 7 and 8.

10. The component of vector $\overline{\mathbf{A}}$ along one of the coordinate axes is the dot product of the relevant unit vector with $\overline{\mathbf{A}}$, e.g.

$$\hat{\mathbf{i}} \cdot \overline{A} = \hat{\mathbf{i}} \cdot (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) = A_x\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + A_y\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + A_z\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = A_x$$

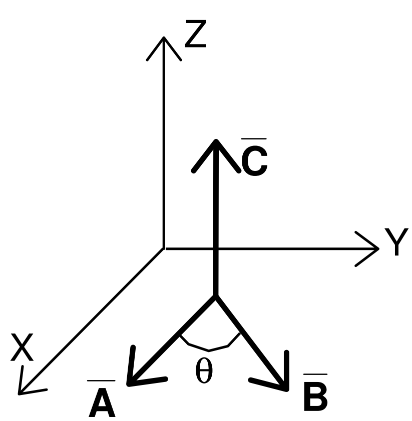
The Cross Product

$\overline{\mathbf{A}} \times \overline{\mathbf{B}} = \overline{\mathbf{C}}$

Magnitude of $\overline{\mathbf{C}}$: $C = AB\sin\theta$

Direction of $\overline{\mathbf{C}}$: $\overline{\mathbf{C}}$ is perpendicular to the plane formed by $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$

Direction is given by the **Right Hand Rule:**



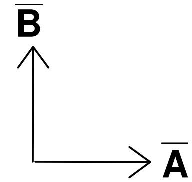
Step 1: Imagine your right hand pointing along $\overline{\mathbf{A}}$
Step 2: Curl the fingers around from $\overline{\mathbf{A}}$ to $\overline{\mathbf{B}}$
Step 3: The thumb then points in the direction of $\overline{\mathbf{C}}$

As drawn above, $\overline{\mathbf{A}} \times \overline{\mathbf{B}} = \overline{\mathbf{C}}$ (direction = +Z)

$\overline{\mathbf{B}} \times \overline{\mathbf{A}} = -\overline{\mathbf{C}}$ (direction = -Z)

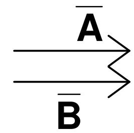
Things to note about the cross product:

1. $\overline{\mathbf{A}} \times \overline{\mathbf{B}}$ is a VECTOR



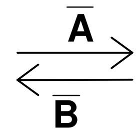
2. If $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are perpendicular $\left|\overline{\mathbf{A}} \times \overline{\mathbf{B}}\right| = AB$

   as $\sin(90^{o}) = 1$



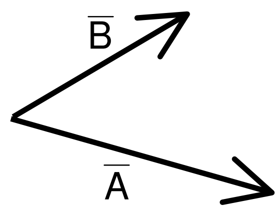
3. If $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are parallel or antiparallel $\left|\overline{\mathbf{A}} \times \overline{\mathbf{B}}\right| = 0$

   as $\sin(0^{o}) = \sin(180^{o}) = 0$



4. The cross product is **not** commutative:

   $\overline{\mathbf{A}} \times \overline{\mathbf{B}}$ is out of the page
   $\overline{\mathbf{B}} \times \overline{\mathbf{A}}$ is into the page
   $\overline{\mathbf{A}} \times \overline{\mathbf{B}} = -(\overline{\mathbf{B}} \times \overline{\mathbf{A}})$

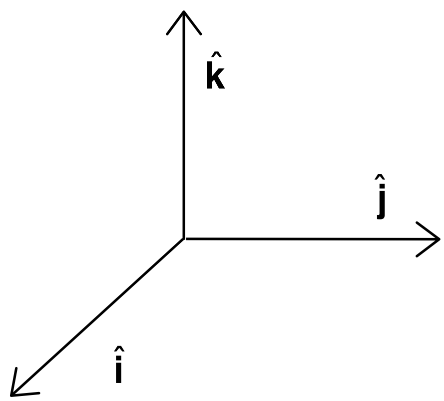


5. For the orthogonal unit vectors:

$$\hat{i} \times \hat{i} = 0 \qquad \hat{i} \times \hat{j} = \hat{k} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k} \qquad \hat{j} \times \hat{j} = 0 \qquad \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{k} \times \hat{j} = -\hat{i} \qquad \hat{k} \times \hat{k} = 0$$



6. The cross product is **not associative**: $\overline{\mathbf{A}} \times (\overline{\mathbf{B}} \times \overline{\mathbf{C}}) \neq (\overline{\mathbf{A}} \times \overline{\mathbf{B}}) \times \overline{\mathbf{C}})$

7. But it **is distributive**: $\overline{\mathbf{A}} \times (\overline{\mathbf{B}} + \overline{\mathbf{C}}) = (\overline{\mathbf{A}} \times \overline{\mathbf{B}}) + (\overline{\mathbf{A}} \times \overline{\mathbf{C}})$

Scalar and vector fields

The value of a scalar or vector quantity often varies with position in space. A function which describes this variation is said to be the FIELD of the quantity.

Scalar field:

A scalar function S(x,y,z) gives the value of S at every point in space.

Examples: S = Height above sea level (geographical contour map)

S = Atmospheric pressure (isobars on a weather map)

Vector field:

A vector function $\bar{\mathbf{F}}$(x,y,z) gives the magnitude and direction of $\bar{\mathbf{F}}$ at every point in space.

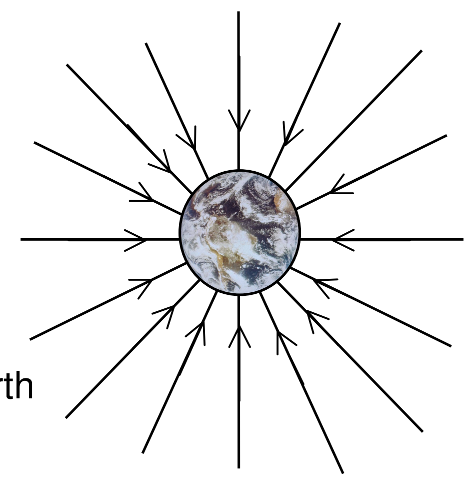
Examples:

1. Wind velocity – indicated by arrows on a weather map
2. The gravitational field of the Earth, $\bar{\mathbf{g}}$:

   Magnitude: $g = \dfrac{GM_E}{R_E^{\ 2}}$

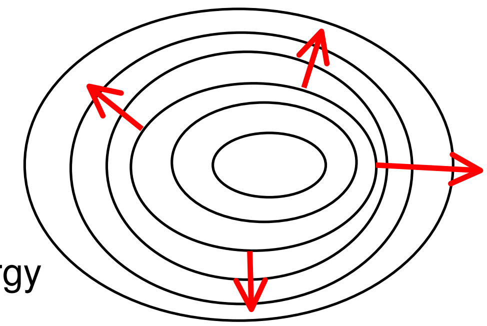
   where M_E = mass of the Earth;
   R_E = Radius of the Earth

   Direction: Towards the centre of the Earth



3. The Electric Field, $\bar{\mathbf{E}}$
4. The Magnetic Field, $\bar{\mathbf{B}}$

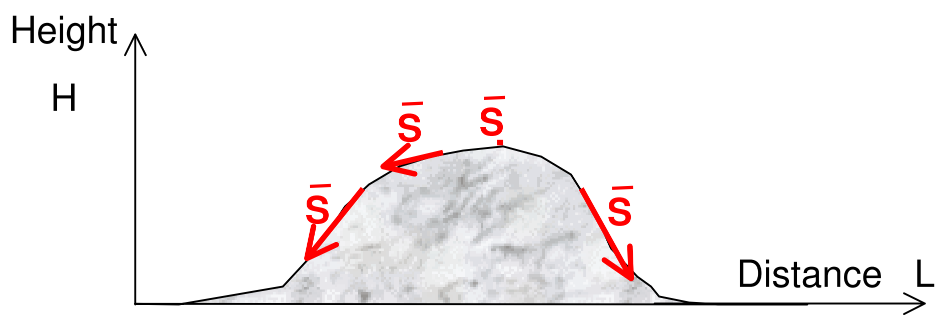
Gradient of a scalar field



Example: H = Height above sea level

Contours = lines of constant height
≡ lines of constant potential energy

Two-dimensional example:



The arrows show the **gradient (slope)** of the hill, $\overline{S}$, at various points. $\overline{S}$ is a vector:

$$\overline{\mathbf{S}} = dH/d\overline{\mathbf{L}}$$

The gradient has: Magnitude (steepness)
and Direction (the direction in which a ball would roll if released at that point)

This is just one example of a general principle:

The gradient of a scalar field is a vector field

If Hx,y,z) is a ANY scalar field, then

$$\mathrm{Grad}(H) = \overline{\nabla} H = \frac{\partial H}{\partial x}\hat{\mathbf{i}} + \frac{\partial H}{\partial y}\hat{\mathbf{j}} + \frac{\partial H}{\partial z}\hat{\mathbf{k}} = \left[\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right] H$$

Grad(H) is also denoted $\overline{\nabla}H$ (pronounced “del”). $\overline{\nabla}$ operates on a scalar field to produce a vector field.

Example: The Electric field, $\overline{\mathbf{E}}$, is the gradient of the Electric Potential, V.