ELECTRIC AND MAGNETIC FIELDS

ANSWERS TO ASSIGNMENT 3

Q1: E = 100 N C^{-1} and points downwards. Cylindrical box has radius 3 m.

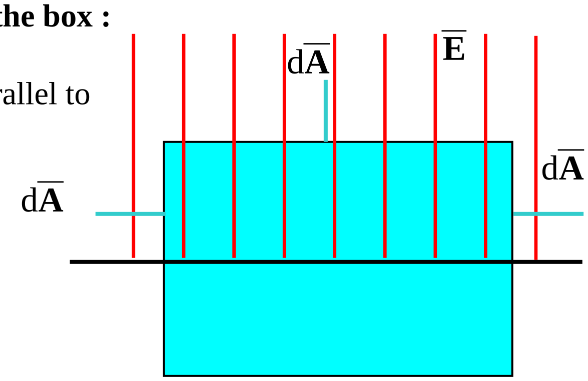
35

Ground ≡ a perfect conductor $\Rightarrow$ Electric field is zero inside the ground
$\Rightarrow$ Electric field lines terminate at negative charges on the surface of the ground.

(i) Total electric flux through the box :

~ 20 for this bit. The answer here is a bit more thourough than is necessary but an understanding of the main steps must be evident.

Deduct 5 marks if the sign is wrong at the end.



- For the top side, $\overline{\mathbf{E}}$ is anti-parallel to the normal vector, so

  $\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = -EdA$

  The magnitude of the field, E, is also the same everywhere over the top side.

- $\overline{\mathbf{E}}$ is zero over the bottom surface
  $\Rightarrow$ no contribution to Φ

- $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are perpendicular for the sides
  $\Rightarrow$ $\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = 0 \Rightarrow$ no contribution to Φ

Therefore

$$\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{\text{top}} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + \int_{\text{bottom}} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + \int_{\text{sides}} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{\text{top}} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + 0 + 0$$

So $$\Phi = \int_{\text{top}} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{\text{top}} EdA = -E\int_{\text{top}} dA = -(E)(\text{Area of top}) = -(100)\pi(3^2) = \mathbf{-2827.43N\ C^{-1}\ m^2}$$

The negative sign signifies electric flux going **into** the box.

(ii) The total enclosed charge is $Q_{enc} = \varepsilon_0\Phi = \mathbf{-2.50 \times 10^{-8}\ C}$ **.**

8

This charge is located at the surface of the ground, where the electric field lines terminate.

(iii) Spherical box of same radius:

The enclosed charge is the same as before since the area of ground inside the box is the same. Since Q_{enc} is the same, the total flux, Q_{enc}/ε_0 is also the same.

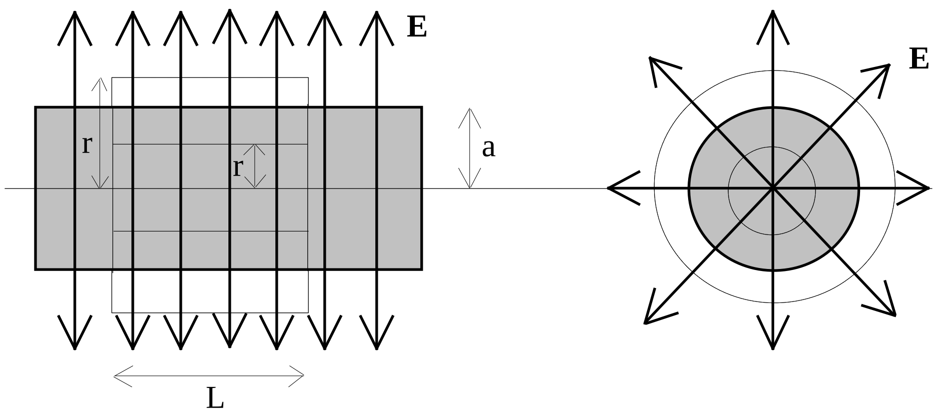
7

Q2: Long plastic (i.e., insulating) rod. Radius a. Uniform charge density inside.

30

Charge per unit length = λ.

(a) By symmetry, the field lines will radiate outwards, pointing perpendicular to the axis of the cylinder.



SIDE VIEW AXIAL VIEW

6 for a good diagram

(b) To use Gauss's Law to find E:

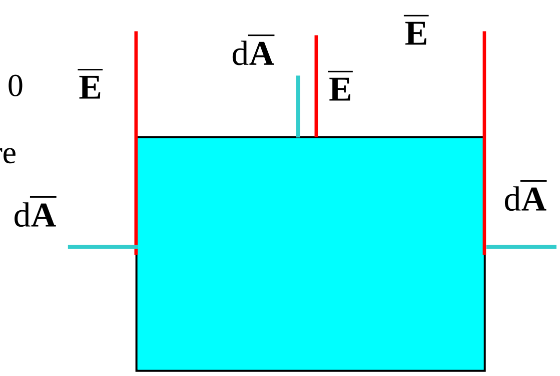
1. Diagram with field pattern: as above

2. Best Gaussian surface: Choose a cylinder, radius r (two cases: r < a and r > a).

For either of the two Gaussian cylinders,

- Flat ends: $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are perpendicular $\Rightarrow \overline{\mathbf{E}}\cdot d\overline{\mathbf{A}} = 0$
- Curved side: $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are parallel

$\Rightarrow \overline{\mathbf{E}}\cdot d\overline{\mathbf{A}} = EdA$



and: because r is the same over the whole of the curved side, so is E.

Step-by-step procedure has been taught explicitly so will probably be followed. But as long as the main ideas are clear it's OK

3. Work out Φ:

$$\Phi = \oint \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \int_{top} \overline{\mathbf{E}}.d\overline{\mathbf{A}} + \int_{bottom} \overline{\mathbf{E}}.d\overline{\mathbf{A}} + \int_{side} \overline{\mathbf{E}}.d\overline{\mathbf{A}} = 0 + 0 + \int_{side} \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \int_{side} EdA = E\int_{side} dA$$

So, $\Phi = (E)(\text{Area of curved surface}) = (2\pi rL)E$

8 for correct Φ and correct derivation

4. Find $Q_{enclosed}$:

(i) r < a: $Q_{enc} = (\text{Charge on length L of rod}) \times \dfrac{\text{Volume of Gaussian cylinder}}{\text{Volume of rod}}$

$\Rightarrow \quad Q_{enc} = (\lambda L)\dfrac{\pi r^2 L}{\pi a^2 L} = \dfrac{r^2}{a^2}\lambda L$

3 + 3 for correct Q_{enc} values

(i) r > a: $Q_{enc} = (\text{Charge on length L of rod}) = \lambda L$

5. Equate Φ and Q_{enc}/ε_0 to find E:

(i) r <a: $(2\pi r L)E = (r^2/a^2)\lambda L/\varepsilon_0 \quad \Rightarrow \quad \boxed{E(r) = \dfrac{\lambda r}{2\pi a^2 \varepsilon_0}}$

3 + 3 for correct final answers

(ii) r >a: $(2\pi r L)E = \lambda L/\varepsilon_0 \quad \Rightarrow \quad \boxed{E(r) = \dfrac{\lambda}{2\pi \varepsilon_0 r}}$

(c) The flux through a cylinder of radius a/2 is

$$\Phi = \oint E.ds = E\oint ds = (E)(Area\ of\ curved\ surface) = \frac{\lambda(a/3)}{2\pi a^2 \varepsilon_0} 2\pi(a/3)(2)$$

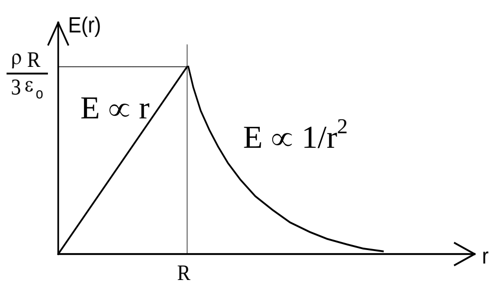
So $\boxed{\Phi = \dfrac{2\lambda}{9\varepsilon_0}}$

4

30

Q3: Given $E = \dfrac{\rho r}{3\varepsilon_0}$ for $r < R$ $\quad E = \dfrac{\rho R^3}{3\varepsilon_0 r^2}$ for $r > R$

(i) Sketch:



6 for a good sketch - doesn't need to be labeled iin as much detail as this.

(ii) $r_1 = 10$ mm, $E_1 = 3.77 \times 10^5$ N C^{-1} $r_2 = 40$ mm, $E_2 = 1.88 \times 10^5$ N C^{-1}

If we determine whether the points are inside or outside, we can use the above formulas to find ρ, R and Q.

If both points are inside, $\dfrac{E_1}{E_2} = \dfrac{r_1}{r_2} \Rightarrow \dfrac{3.77}{1.88} = \dfrac{10}{40} \Rightarrow 2 = 0.25$ (NOT TRUE)

If both are outside, $\dfrac{E_1}{E_2} = \dfrac{r_2^{\,2}}{r_1^{\,2}} \Rightarrow \dfrac{3.77}{1.88} = \dfrac{40^2}{10^2} \Rightarrow 2 = 8$ (NOT TRUE)

Therefore the only possibility is that r_1 is inside and r_2 is outside.

3 for actually proving this

(a) r_1 inside $\Rightarrow \rho(0.010)/(3\varepsilon_0) = 3.77 \times 10^5 \Rightarrow$ **$\rho = 1.0 \times 10^{-3}$ C m^{-3}** 5

(b) Therefore, r_2 outside $\Rightarrow \dfrac{(1.0\times10^{-3})R^3}{3\varepsilon_0(0.040)^2} = 1.88\times10^5 \Rightarrow$ **R = 20 mm** 5

(c) $Q = \dfrac{4}{3}\pi R^3 \rho \Rightarrow$ **$Q = 3.35 \times 10^{-8}$ C** 5

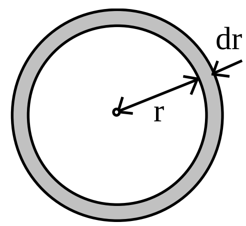
5 **Q4:** $\rho = \dfrac{e}{8\pi b^3}\exp\left[-\dfrac{r}{b}\right]$ C m^3 where b = 2.3 x 10^{-16} m

Follow the procedure for using Gauss's law using a spherical Gaussian surface of radius r.

Steps 1 - 3 give $\Phi = (4\pi r^2)E$ where E is the magnitude of the field at r.

Step 4: Find $Q_{enclosed}$:

Define a thin spherical shell, radius r, thickness dr:



The charge contained in the shell is

dq = ρ(r)(volume of shell)

So, $dq = \dfrac{e}{8\pi b^3}\exp\left[-\dfrac{r}{b}\right]\left[4\pi r^2 dr\right] = \dfrac{er^2}{2b^3}\exp\left[-\dfrac{r}{b}\right]dr$

The total charge inside a Gaussian sphere of radius r is therefore

$$q(r) = \frac{e}{2b^3}\int_0^r r^2 \exp\left[-\frac{r}{b}\right]dr$$

The standard integral, with x=r/b gives

$$Q_{enc} = \frac{e}{2}\int_0^{r/b} x^2\exp[-x]dx = \frac{e}{2}\left[-x^2e^{-x} - 2e^{-x}(x+1)\right]_0^{r/b}$$

So $$Q_{enc} = \frac{e}{2}\left[2 - \left(\frac{r}{b}\right)^2 e^{-r/b} - 2e^{-r/b}\left(\frac{r}{b}+1\right)\right]$$

Putting $\Phi = Q_{enc}/\varepsilon_0$ gives $$E_r = \frac{e}{8\pi\varepsilon_0 r^2}\left[2 - \frac{r^2}{b^2}\exp\left(-\frac{r}{b}\right) - 2\left(\frac{r}{b}+1\right)\exp\left(-\frac{r}{b}\right)\right]$$

Putting in the numerical values gives E = 2.13 x 10^{21} N C^{-1} at r = 5 x 10^{-16} m

This is 2.7 times smaller than the value we would get if we regarded the proton as a point charge, (E = 5.67 x 10^{21} N C^{-1}) .

2 for applying Gauss's Law and using correct volume element etc.

2 for coping with the integration

0.5

0.5

Round up any half marks at the end