Permutations, Strings and Feynman Graphs.

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Based on papers

From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group
arXiv:1002.1634[hep-th] ; Robert de Mello Koch, Sanjaye Ramgoolam
Permutations, Strings and Feynman Graphs
Robert de Mello Koch, Sanjaye Ramgoolam ; to appear : arXiv:1109.****

Introduction

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Example $\sigma_1, \sigma_2, \sigma_3$ are 3 permutations among the 6 in S_3.

$$\begin{aligned}
\sigma_1 &: \{1,2,3\} \to \{1,2,3\} \\
\sigma_2 &: \{1,2,3\} \to \{2,1,3\} \\
\sigma_3 &: \{1,2,3\} \to \{2,3,1\}
\end{aligned}$$

Other ways of describing σ_2 :

$$\begin{aligned} \sigma_2(1) &= 2, \sigma_2(2) = 1, \sigma_2(3) = 3 \\ \sigma_2 &= 213 \\ \sigma_2 &= (12)(3) \end{aligned}$$

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$$\begin{aligned}\sigma_1 &= (1)(2)(3) \\ \sigma_2 &= (12)(3) \\ \sigma_3 &= (123)\end{aligned}$$

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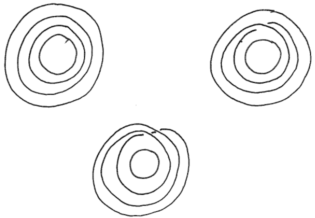
Two permutations σ, σ' having the same cycle structure are related by conjugaction .

$$\sigma = \gamma \sigma \gamma^{-1}$$

e.g (12)(3) and (13)(2) are related by $\gamma = (1)(23)$.

S_3 has 3 cycle structures, equivalently 3 conjugacy classes.
Consider the possible states of 3-strings winding around a circle.

Fig. 1



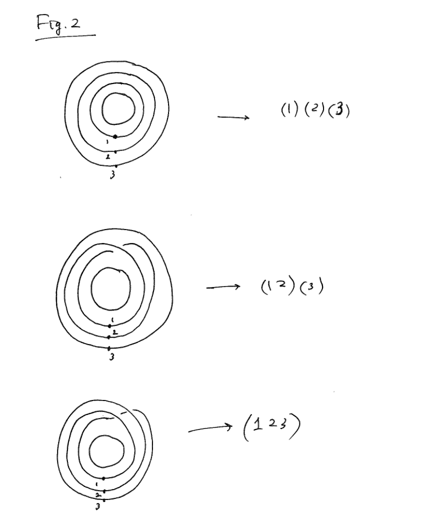
$X(\sigma) = \sigma$ Singly Wound

$X(\sigma) = 2\sigma$ Doubly Wound

$X(\sigma) = 3\sigma$ Triply Wound

$$X \sim X + (2\pi)$$
$$\sigma \sim \sigma + (2\pi)$$

Starting from a configuration of such strings we can label the points above a fixed spacetime point and obtain a permutation.



The number of winding states, or cycle structures in S_n, is the number of partitions of n, called $p(n)$: a well-studied number in Mathematics. Its asymptotics is relevant to Hagedorn transition.

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When we consider string interactions, the permutations themselves matter, not just the cycle structure they belong to.

The connection between strings and permutations plays a central role in gauge-string duality. I will explain 3 connections to illustrate this.

- Large N expansion of 2 dimensional Yang Mills partition function.
- Large N expansion of Hermitian Matrix model correlators.
- Feynman Graph counting in scalar field theory.

The third example suggests that Large N is not crucial to strings emerging from Quantum Field theory.

Gauge-String Duality.
Some dynamics of quantum field theory with matrix fields has a dual description in terms of String theory. Maldacena's AdS/CFT duality between strings on $AdS_5 \times S^5$ and $N = 4$ SYM with $U(N)$ gauge group is one of the richest examples.

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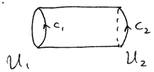
In lower dimensions, we have $2D$ pure Yang Mills theory with $U(N)$ gauge group, on Riemann surface Σ_G of genus G and area A.

This theory is quasi-topological and the partition function depends only on G, A. In this case, the string theory has 2D target Σ_G, as discovered by Gross and Taylor in mid-nineties.

Permutations are key to organising the gauge invariant operators. For 2dYM on a cyclinder, defining the partition function requires specifying the boundary condition, which is a group element U in $U(N)$ at each boundary.

Fig. 3 :



$$\mathcal{U}_1 = e^{i\oint_{c_1}\vec{A}\cdot d\vec{s}}$$

$$U_2 = e^{i\oint_{c_2}\vec{A}\cdot d\vec{s}}$$

The gauge-invariant functions are traces.

$$tr(U^3), trU^2 trU, (trU)^3$$

Some are linear in traces, some non-linear. Permutations give a unified linear way way of thinking about all of them.

$$
\begin{aligned}
trU^2 &= U^{i_1}_{i_2} U^{i_2}_{i_1} \\
&= U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}} \\
\text{with } \sigma &= (12)
\end{aligned}
$$

$$
\begin{aligned}
(trU)^2 &= U^{i_1}_{i_1} U^{i_2}_{i_2} \\
&= U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}}
\end{aligned}
$$

Multi-traces are constructed by using different permutations.

$$
tr_{V^{\otimes n}}(\sigma U^{\otimes n})
$$

Different permutations with the same cycle structure give the same trace. Replcing $\sigma \rightarrow \gamma\sigma\gamma^{-1}$ leaves the trace invariant.

In 2dYM, the partition function $Z(U_1, U_2)$ on a cylinder (and any Riemann surface) can be written exactly in terms of representations of $U(N)$.

We can transform to a permutation basis

$$Z(\sigma_1, \sigma_2) = \int dU_1 dU_2 Z(U_1, U_2) tr_n(\sigma_1 U_1^\dagger) tr_n(\sigma_2 U_2^\dagger)$$

$$Z(\sigma_1, \sigma_2) = \sum_{\gamma \in S_n} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$

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This is the answer in the zero area limit.

The δ function is defined like a Kronecker delta, except over the symmetric group :

$$\begin{aligned}\delta(\sigma) &= \text{1 if } \sigma = \text{ identity}\\ &= \text{0 otherwise}\end{aligned}$$

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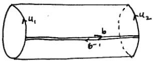
The permutation γ is the re-lebelling of sheets of the cover in going from one boundary to another. The delta function ensures that the two cycle structures are the same.



$u_1 b u_2 b^{-1} = 1$ in π_1(Cylinder)

Figure: Paths and permutations on cylinder

At non-zero area the sum is modified to include additional permutations which can be interpreted as a counting of branched covers (holomorphic maps) where $\partial_z f$ of the map is allowed to vanish at certain points on the worldsheet.

Let us leave 2dYM aside, to give a simple illustration of this point in the Hermitian matrix model.

The physics of the Gaussian measure for the eigenvalue distribution is encoded in correlators of traces :

$$\int dX \;\; e^{-\frac{1}{2} trX^2} \mathcal{O}(X)$$

The observables are general traces :

$$\begin{array}{l} trX \\ trX^2, trXtrX \\ trX^3, trX^2trX, trXtrXtrX \end{array}$$

One finds that

$$< \mathcal{O}_p >= \frac{1}{n!} \sum_{\sigma_1 \in p \in S_n} \sum_{\sigma_2 \in [2^{n/2}]} \sum_{\sigma_3} \delta(\sigma_1\sigma_2\sigma_3) N^{C_{\sigma_3}}$$

Here n is even. The permutation σ_3 is arbitrary, but σ_2 has the cycle structure $[2^{n/2}]$, i.e of type

$$(12)(34)\cdots(n-1\ n)$$

This comes from the fact that the computation of the correlators can be done by Wick contractions, which are pairings of the n matrices in the observable.

One can use some classical mathematics of Riemann to relate this formula directly to the geometry of branched covers, i.e holomorphic maps a worldsheet Σ to a sphere.

Holomorphy :

$$\partial_{\bar{z}} f = 0$$

The powers of N keep track of the genus of the worldsheet.

Three permutations : If $\partial_z f(P) = 0$ then $f(P) \in \{0, 1, \infty\}$.

Details in arXiv:1010.1634

A third connection involves a QFT without large N. Just real scalar field theory, for concreteness, take vacuum diagrams in ϕ^4 theory.

Calculations in QFT are simplified by organizing the large number of Wick contractions, into graphs, each of which comes with a symmetry factor.

For $v = 1$ there is one graph. For $v = 2$, there are 3 graphs, etc.

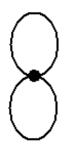


Figure: One vertex vacuum diagram in ϕ^4 theory

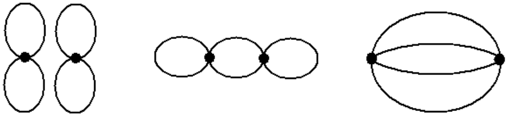


Figure: Two vertex vacuum diagrams in ϕ^4 theory

This sequence of vacuum diagrams

$$1, 3, 7, 20, 56, 187, 654, 2705, 12587, 67902, 417065, ..$$

has an expresion in terms of string amplitudes, of the kind that appears in 2dYM.

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Number of diagrams with v vertices

$$= \frac{1}{|H_1||H_2|} \sum_{\sigma_1 \in H_1 \in S_{4v}} \sum_{\sigma_2 \in H_2 \in S_{4v}} \sum_{\gamma \in S_{4v}} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$

H_1 is a subgroup of S_{4v} :

$$(S_4 \times S_4 \cdots \times S_4) \ltimes S_v \equiv S_v[S_4]$$

There are v copies of S_4 and S_v acts as an automorphism of this product group.

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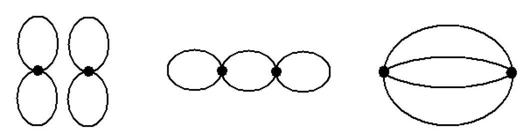
H_1 is the symmetry of the v 4-valent vertices. H_2 is the subgroup of permutations which commute with

$$(12)(34)\cdots(4v-1\;4v)$$

which has to do with the pairing-property of Wick contractions.

The key step in deriving this expression is to describe the graph in terms of a pair of data Σ_0, Σ_1, where Σ_0 is associated with vertices and Σ_1 with Wick contractions



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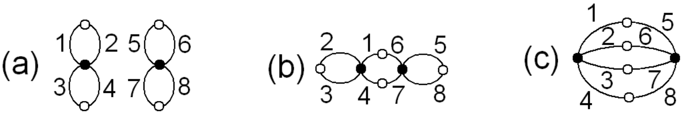

Figure: Numbering the half-edges

$$
\begin{array}{lrcl}
(a) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (12)(34)(56)(78) \\
 & & & \\
(b) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (23)(16)(47)(58) \\
 & & & \\
(c) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (15)(26)(37)(48)
\end{array}
$$

This also leads to neat symmetric group expressions for symmetry factors which have a string interpretation.

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Are there physical versions of such dualities involving non-trivial dependence on space-time and momenta ?