Permutations, Strings and Feynman Graphs.

Sanjaye Ramgoolam

Queen Mary - London

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Based on papers Robert de Mello Koch, Sanjaye Ramgoolam, "From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group " arXiv:1002.1634[hep-th]
Strings from Counting Feynman Graphs : without large N
Robert de Mello Koch, Sanjaye Ramgoolam ; to appear : arXiv:1110.****
+ Earlier ...

Introduction

Feynman Graphs are used to compute correlators in Quantum Field Theory (QFT) .

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Quantum chromodynamics requires non-abelian gauge symmetry $SU(3)$. 't Hooft used the idea of large N gauge theory e.g $U(N)$ local symmetry, in a $1/N$ expansion, as an approximation.

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He interpreted the Riemann surfaces in terms of string theory as string worldsheets of a dual string theory.

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Early-Mid 90's : concrete combinatoric versions of gauge-string dualities were found. Precise connections to holomorphic map counting.

Mid-late 90's : AdS/CFT – deeper versions of gauge-string duality, where observables have non-trivial space-time dependences.

All relies on large N.

At the combinatoric level, there are also string dualities for Feynman graph counting which do not rely on large N, and are relevant to scalar field theories or QED.

OUTLINE

- Review of strings and simple examples of gauge-string duality.
- Two dimensional Yang Mills at large N and TFT with S_n gauge group.
- Schur-Weyl duality ; Riemann existence theorem.
- Recent work : Graph counting in scalar field theories and string amplitudes. Without large N.
- Connections between strings, permutations, graphs hinting at more general QFT-string dualities

String theory :

Point particles moving in spacetime are replaced by one-dimensional objects.

Worldlines - curves describing trajectories of points in space-time - are replaced by Worldsheets (two-dimensional surfaces).

A string theory is a path-integral over spaces of maps from worldsheets to space-time.

Worldsheets have a genus h. A string theory has a parameter g_{st}, the *string coupling* .

Contributions to physical observables, from genus h world-sheets, are weighted by g_{st}^{2h-2}

Traditionally string theories are described by writing a worldsheet action

$$S = \int_{\Sigma_h} \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

X^μ are space-time coordinates. They are dynamical variables $X^\mu(\sigma, \tau)$ depending on world-sheet coordinates (σ, τ).

This traditional approach leads to integrals over $\mathcal{M}_{h,n}$, the moduli space of conformal structures of the world-sheet metric g

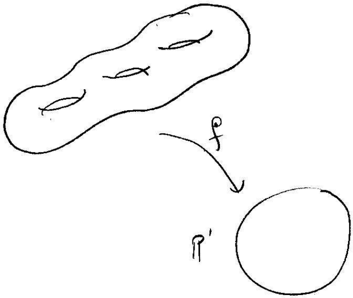
A recent theme in last 20 years :

Emergent string theories.

Simplest string theories emerge from Matrix integrals.

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2}trX^2}\mathcal{O}_p(X)$$

X is an $N \times N$ hermitian matrix. $\mathcal{O}_p(X)$ is a product of traces $trX, trX^2, \cdots$. For fixed degree d in X, the observables are parametrized by partitions p of d.



$$\bar{\partial} f = 0$$

Permutation sums

$$\langle \mathcal{O}_p \rangle = \frac{1}{(2n)!} \sum_{\sigma \in p} \sum_{\gamma \in [2^n]} \sum_{\tau \in S_{2n}} \delta(\sigma\gamma\tau) N^{C_\sigma + C_\tau - n}$$

The sum γ is over the conjugacy class $[2^n]$ – of permutations with n cycles of length 2.

where

$$\begin{aligned} \delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Branched Covers

Sum of three permutations with a constraint that they multiply to 1. Weighted by $1/N$ according to cycle structures.

$\pi_1(\mathbb{P}^1 \setminus \{3\text{punctures}\})$ is the group with 3 generators and one relation.

From covering space theory, the permutation sum is counting branched covers of $\mathbb{P}^1$.

Holomorphic maps

Branched at 3 points. If $\partial_z f(P) = 0$ for some P on the cover, then $f(P)$ is one of three points.

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Branched covers are holomorphic maps.

The string theory thus emerging is simplest at large N. The contributions from genus h are weighted by N^{2-2h}.

We expect there is a string theory where $g_{st} = 1/N$ and the worldsheet path integral over maps which localizes to holomorphic maps – and which computes the same observables $\mathcal{O}_p$.

Integrals $\longrightarrow$ QFT

In Quantum field theories the matrix X is replaced by a space-time dependent matrix – called a field.

$$X \longrightarrow X(t, x_1, x_2, x_3)$$

The integrand is replaced by some spacetime integral e.g.

$$e^{\frac{1}{2}\int d^4x tr \partial_\mu X \partial^\mu X - trX^2}$$

The integral $\int dX$ becomes a path integral.

The most famous example is four dimensional QFT, with $U(N)$ gauge group (and $N = 4$ supersymmetry).

The fields appearing in the action are connections for $U(N)$ bundles over $\mathbb{R}^4$ (along with other fields).

$$X(x_1, x_2, x_3, t) \to A(x_1, x_2, x_3, t)$$

A is a 1-form with values in the Lie algebra $u(N)$. Action is constructed from the curvature of this connection.

In this case the dual string theory is *conjectured* to be ten-dimensional, and the space-time is $AdS_5 \times S^5$. This version of string-QFT duality is called AdS/CFT.

There is a lot of evidence for the conjecture. There are physical arguments based on D-brane physics and string theory in favour of the conjecture.

Simpler models of gauge-string duality were precursors of AdS/CFT, and serve to get insights on the mathematical mechanisms for gauge-string duality.

Another lower dimensional QFT will be of interest. It is $U(N)$ gauge theory on a closed Riemann surface Σ_G, or with boundaries $\Sigma_{G,B}$

The case where the Riemann surface is a cylinder is of particular interest.

The path integral for a closed Riemann surface depends only on the area A and genus G, and of course N. Take $G > 1$.

The leading large N answer is

$$Z(G, A = 0) = \sum_{s_1, s_2, \cdots s_G} \sum_{t_1, t_2, \cdots t_G} \delta(s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2^{-1} \cdots s_G t_G s_G^{-1} t_G^{-1})$$

where

$$\begin{aligned} \delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

This is counting homomorphisms from $\pi_1(\Sigma_G)$ to S_n.

From Riemann-existence theorem, this is equivalent to counting unbranched covers of $\Sigma_h \to \Sigma_G$.

At higher orders in the $1/N$ expansion, the coefficients have an interpretation in terms of branched covers of Σ_G.

The large N answer can also be interpreted in terms of gauge theory with S_n gauge group.

In lattice gauge theory, we discretize (triangulate) the surface and associate group elements to edges. To the 2-cells (triangles) we associate a weight depending on the product σ of group elements for edges around the 2-cell.

The continuum limit is obtained by refining the lattice and studying the resulting change of the lattice action.

The simplest lattice gauge theory for S_n is defined using

$$Z_P(\sigma) = \delta(\sigma) = \sum_R \frac{d_R \chi_R(\sigma)}{n!}$$

Under refinement of the discretization, the weight of for 2-cells is unchanged.

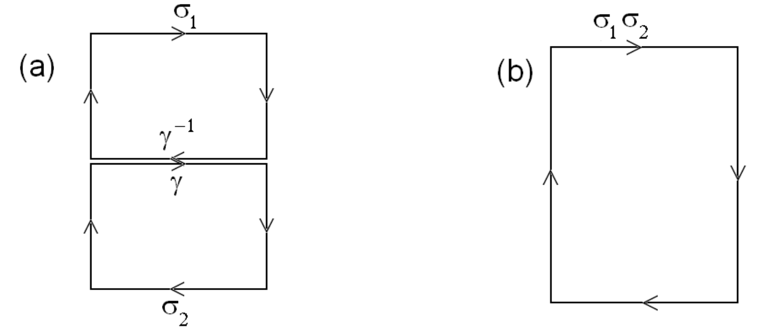
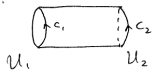


Figure: Two discretizations of the disc with the same boundary condition.

The continuum result can be computed with a single-cell. It leads to the answer given before. The lattice gauge theory is actually a topological field theory.

For 2dYM on a cyclinder, defining the partition function requires specifying the boundary condition, which is a group element U in $U(N)$ at each boundary.

Fig. 3 :



$$\mathcal{U}_1 = e^{i\oint_{C_1}\vec{A}\cdot d\vec{s}}$$

$$U_2 = e^{i\oint_{C_2}\vec{A}\cdot d\vec{s}}$$

The result is a gauge-invariant function of U. Can be written in terms of characters of U in irreps. of $U(N)$. Instructive to go transform to a permutation basis of these gauge invariant operators.

Gauge-invariant functions are traces.

$$tr(U^3), trU^2trU, (trU)^3$$

$$
\begin{aligned}
trU^2 &= U^{i_1}_{i_2} U^{i_2}_{i_1} \\
&= U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}} \\
\text{with } \sigma &= (12)
\end{aligned}
$$

$$
\begin{aligned}
(trU)^2 &= U^{i_1}_{i_1} U^{i_2}_{i_2} \\
&= U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}}
\end{aligned}
$$

Multi-traces are constructed by using different permutations.

$$tr_{V^{\otimes n}}(\sigma U^{\otimes n})$$

Different permutations with the same cycle structure give the same trace. Replcing $\sigma \to \gamma\sigma\gamma^{-1}$ leaves the trace invariant.

In 2dYM, the partition function $Z(U_1, U_2)$ on a cylinder (and any Riemann surface) can be written exactly in terms of representations of $U(N)$.

We can transform to a permutation basis

$$Z(\sigma_1, \sigma_2) \;=\; \int dU_1 dU_2 Z(U_1, U_2) tr_n(\sigma_1 U_1^\dagger) tr_n(\sigma_2 U_2^\dagger)$$

$$Z(\sigma_1, \sigma_2) = \sum_{\gamma \in S_n} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$

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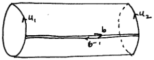
This is the answer in the zero area limit.



$u_1 b u_2 b^{-1} = 1$ in π_1(Cylinder)

Figure: Paths and permutations on cylinder

The partition function vanishes unless σ_1 and σ_2 are in the same conjugacy class.

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Physical interpretation : The covers are string worldsheets.

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Physical interpretation : The covers are string worldsheets.

The inverse image of a boundary circle is a union of circles, with winding numbers described by the permutations σ_1, σ_2.

The condition $\sigma_1 \gamma \sigma_2 \gamma^{-1}$ says that that there are no branch points in the interior of the cylinder, so up to relabellingby γ, the windings are the same.

A third connection involves a QFT without large N. Just real scalar field theory, for concreteness, take vacuum Feynman graphs in ϕ^4 theory.

Calculations in QFT are simplified by organizing the large number of Wick contractions, into graphs, each of which comes with a symmetry factor.

Each Feynman graph is associated with an integral. Not of interest at the moment. Multiplicities and Symmetry factors.

For $v = 1$ there is one graph. For $v = 2$, there are 3 graphs, etc.

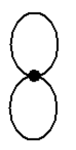

Figure: One vertex vacuum diagram in ϕ^4 theory

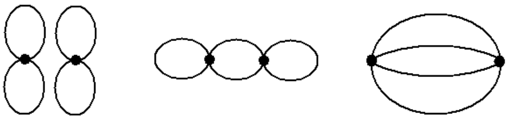

Figure: Two vertex vacuum diagrams in ϕ^4 theory

This sequence of vacuum diagrams

$$1, 3, 7, 20, 56, 187, 654, 2705, 12587, 67902, 417065, ..$$

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Number of diagrams with v vertices

$$= \frac{1}{|H_1||H_2|} \sum_{\sigma_1 \in H_1 \in S_{4v}} \sum_{\sigma_2 \in H_2 \in S_{4v}} \sum_{\gamma \in S_{4v}} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$

A well-defined quantity in S_n TFT .

H_1 is a subgroup of S_{4v} :

$$(S_4 \times S_4 \cdots \times S_4) \ltimes S_v \equiv S_v[S_4]$$

There are v copies of S_4 and S_v acts as an automorphism of this product group.

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H_2 is a subgroup of S_{4v} :

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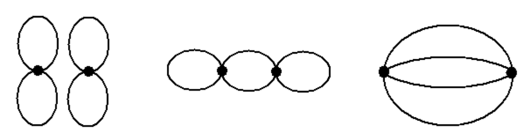
H_1 is the symmetry of the v 4-valent vertices. H_2 is the subgroup of permutations which commute with

$$(12)(34) \cdots (4v - 1\ 4v)$$

which has to do with the pairing-property of Wick contractions.

The key step in deriving this expression is to describe the graph in terms of a pair of data Σ_0, Σ_1, where Σ_0 is associated with vertices and Σ_1 with Wick contractions



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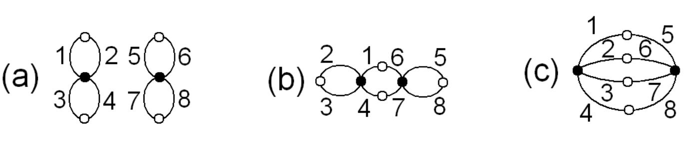

Figure: Numbering the half-edges

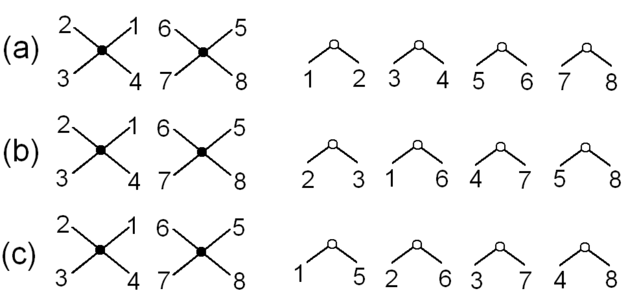

Figure: Splitting the half-edges

$$
\begin{array}{lrcl}
(a) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (12)(34)(56)(78) \\
 & & & \\
(b) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (23)(16)(47)(58) \\
 & & & \\
(c) & \Sigma_0 & = & <1,2,3,4><5,6,7,8> \\
 & \Sigma_1 & = & (15)(26)(37)(48)
\end{array}
$$

This also leads to neat symmetric group expressions for symmetry factors which have a string interpretation.

The pair Σ_0, Σ_1 can be understood in terms of a double coset description of Feynman Graph counting :

$$S_{4v} \setminus (S_{4v} \times S_{4v})/(H_1 \times H_2)$$

which leads directly to counting formulae for the Feynman graphs in terms of cycle indices of H_1, H_2. (classic results of Read in 1960's)

which are here interpreted in terms of S_n TFT and covering spaces.

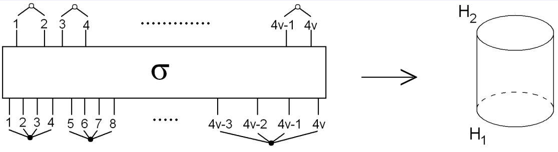

Figure: Double coset connection

This counting can be expressed in terms of representation theory

$$\sum_{R \in Rep(S_{4v})} \mathcal{M}^{R}_{1_{H_1}} \mathcal{M}^{R}_{1_{H_2}}$$

Similar results can be derived for a variety of Feynman graph problems.

Group	acts on	Feynman Graph problem
$S_v[S_4]$	permutations in $[2^{2v}]$ of S_{2v}	ϕ^4 theory
$S_v[S_3]$	permutations in $[2^{\frac{3v}{2}}]$ of $S_{3v/2}$	ϕ^3 theory
$S_{v_4}[S_4] \times S_{v_3}[S_3]$	permutations in $[2^{\frac{(3v_3+4v_4)}{2}}]$ of $S_{3v_3+4v_4}$	$\phi^3 + \phi^4$ theory
$S_v[S_2]$	All permutations in S_{2v}	Yukawa/QED
$S_v[S_2]$	All even-cycle permutations in S_{2v}	Furry QED
$S_v[\mathbb{Z}_4]$	permutations in $[2^{2v}]$ of S_{2v}	Large N expansion of Matrix ϕ^4

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There are “topological” dualities between QFT and strings related to the counting of Feynman diagrams which do not involve large N.

Are there physical versions of such dualities involving non-trivial dependence on space-time and momenta ?