ELECTROMAGNETIC INDUCTION
(*Y&F* Chapters 30, 31; *Ohanian* Chapter 32)

This handout covers:

- Motional emf and the electric generator
- Electromagnetic Induction and Faraday's Law
- Lenz's Law
- Induced electric field
- Inductance
- Magnetic energy

The Electric and magnetic fields are inter-related

The electric and magnetic fields are not independent. In fact:

- A changing magnetic field induces an electric field
- The reverse is also true: a changing electric field induces a magnetic field

To see how this occurs, we first consider MOTIONAL emf

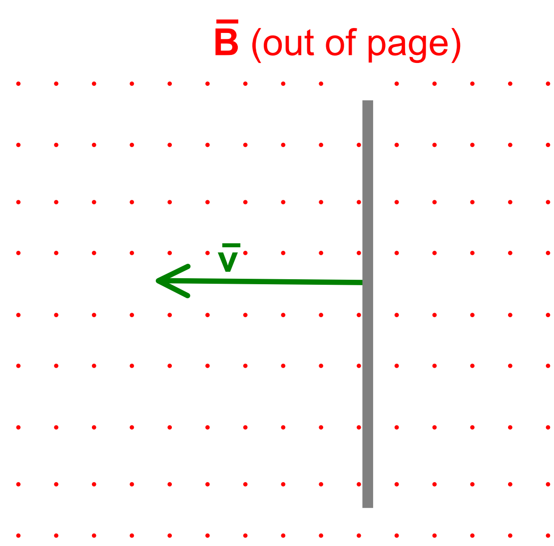
Motional emf

Consider a wire moving with velocity **v** in a magnetic field $\overline{\mathbf{B}}$.

Assume for simplicity that $\overline{\mathbf{v}}$ and $\overline{\mathbf{B}}$ are perpendicular

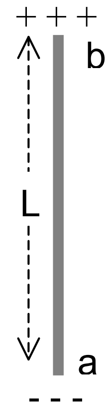
The electrons in the wire experience a force

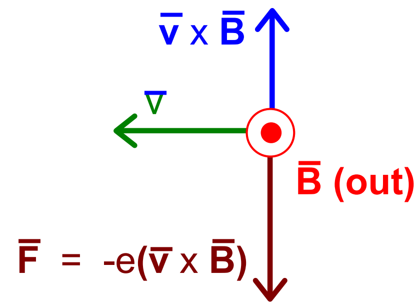
$$\overline{\mathbf{F}} = -\mathbf{e}(\overline{\mathbf{v}} \times \overline{\mathbf{B}})$$



This produces a separation of charge with an excess of negative charge at one end and positive charge at the other.

There is therefore an INDUCED emf between the two ends.





Induced emf $\mathcal{E}$ = work done to move a unit positive charge from a to b against the magnetic force

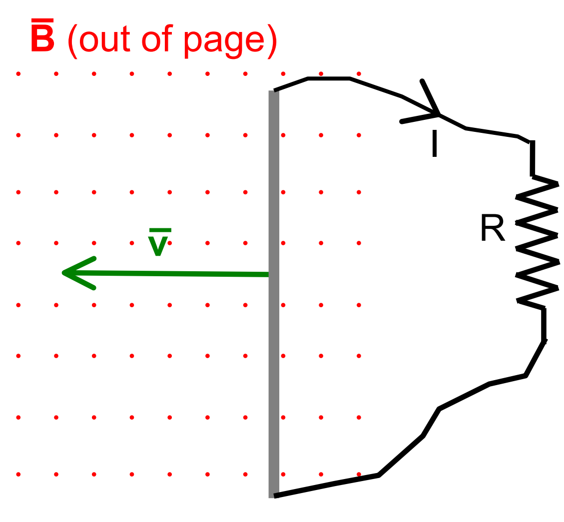
$$F = QvB = vB$$

Work done = (Force)(Distance) $\Rightarrow$ $\mathcal{E} = vBL$ MOTIONAL emf

The electric generator

Assume that the ends of the wire are connected up through some external circuit (which we represent here by a simple resistor).

Current I flows and dissipates electrical power in R.



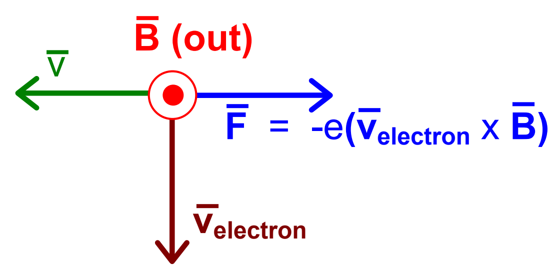
Where does this power come from?

The current is due to electrons moving in the wire with some drift velocity $\mathbf{v}_{electron}$.

The electrons therefore experience a magnetic force

$$\bar{\mathbf{F}} = -\mathbf{e}(\bar{\mathbf{v}}_{electron} \times \bar{\mathbf{B}})$$

This force OPPOSES the motion of the wire through the field.



The overall force on the wire is $F_{mag} = \Sigma F$ for all the electrons.

Recall: Force on a current-carrying wire in a magnetic field is

$F_{mag} = BIL$.

So, the power delivered to the circuit comes from the effort needed to PUSH the wire through the magnetic field against this force.

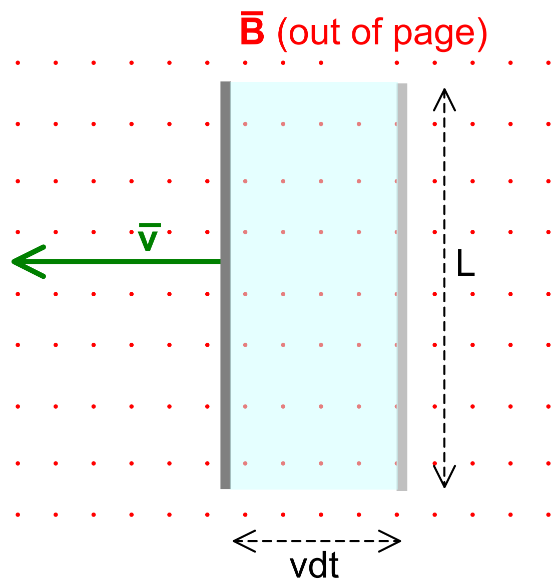
The magnetic field thus acts as a mediator in the conversion of MECHANICAL ENERGY into ELECTRICAL ENERGY.

This is the principle of the ELECTRIC GENERATOR.

Exercise: Show that the power needed to move the wire through the magnetic field is equal to the power dissipated in the resistor.

Motional emf and magnetic flux

Recall: $\mathcal{E} = vBL$



In a time dt, the conductor moves a distance vdt.

The area swept out is therefore

$ds = Lvdt$

The magnetic flux passing through this area is

$d\Psi = Bds = BLvdt$

Therefore $$\frac{d\Psi}{dt} = vBL \Rightarrow \mathcal{E} = \frac{d\Psi}{dt}$$

The induced emf is equal to the rate of sweeping out of magnetic flux

Note: 1. This applies to ANY shape of conductor in ANY magnetic field.

2. The same equation applies to a stationary conductor in a changing magnetic field.

Faraday's law of induction

The induced emf around any closed path in a magnetic field is equal to minus the rate of change of the magnetic flux intercepted by the path:

$$\mathcal{E} = -\frac{d\Psi}{dt}$$

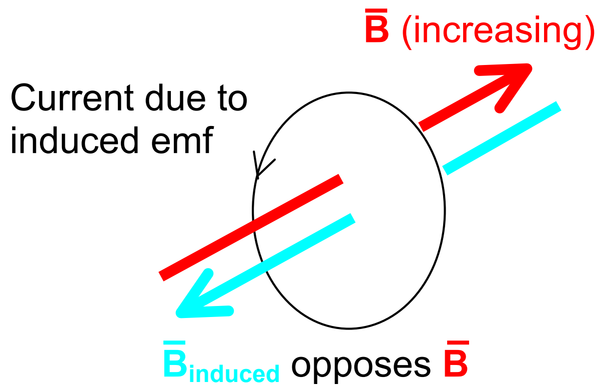
Why the negative sign?

Lenz's law

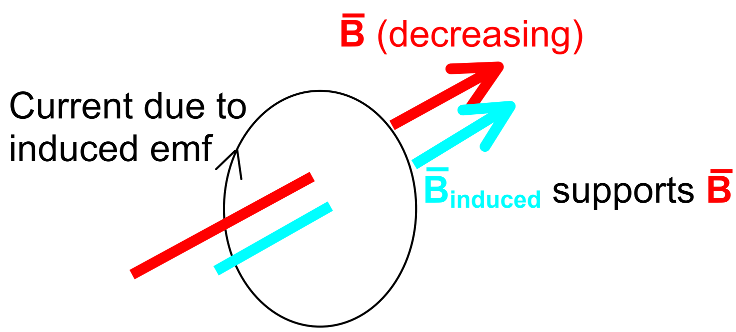
This is represented by the negative sign in the above equation for the induced emf.

The induced emf is in such a direction as to OPPOSE the change in magnetic field that produces it.

If ψ is **increasing** then $\mathcal{E}$ causes current to flow so as to generate a magnetic field which OPPOSES ψ.



If ψ is **decreasing** then $\mathcal{E}$ causes current to flow so as to generate a magnetic field which SUPPORTS ψ.



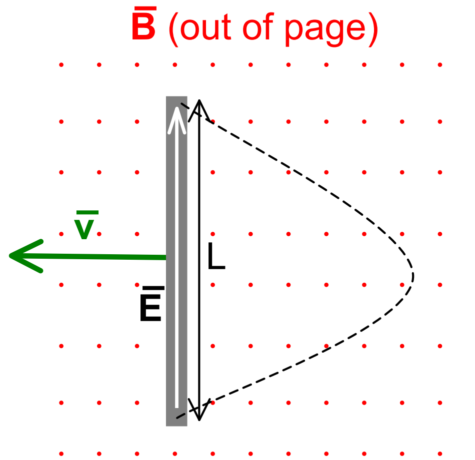
Induced electric field

Consider again a wire moving in a magnetic field.

Look at the situation from the point of view of someone who moves along with the wire:

v = 0 $\Rightarrow$ there is no magnetic force

Therefore, this observer interprets the force acting on the electrons in the wire as being due to an INDUCED ELECTRIC FIELD.



$\Rightarrow$ $\oint \bar{\mathbf{E}} \cdot d\bar{\mathbf{L}}$ around the closed path shown $\neq$ 0.

$\Rightarrow$ The induced electric field is not conservative.

Faraday's Law is usually written in this form:

$$\oint \bar{\mathbf{E}} \cdot d\bar{\mathbf{L}} = -\frac{d\Psi}{dt}$$

FARADAY'S LAW

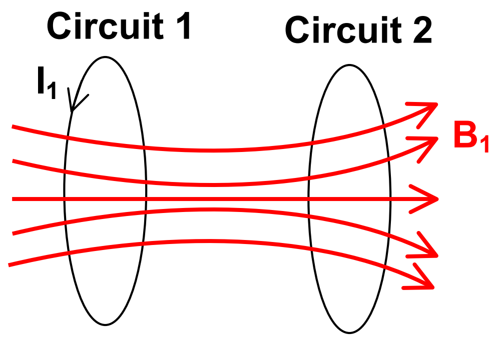
MAXWELL'S 3rd EQUATION

In words: The line integral of the electric field around a closed path is equal to minus the rate of change of magnetic flux through the path

Mutual inductance

Faraday's Law $\Rightarrow$ a changing $\overline{\mathbf{B}}$ $\rightarrow$ induced emf

Consider two nearby circuits:



Changing current in Circuit 1

$\downarrow$

Changing magnetic field through Circuit 2

$\downarrow$

Induced emf in Circuit 2

$\downarrow$

Current flows in Circuit 2

Let $I_1(t)$ create magnetic field $B_1(t)$.
Let ψ_{21} be the flux through circuit 2 due to I_1

Clearly, $B_1 \propto I_1$, so $\psi_{21} \propto I_1$

DEFINITION: The constant of proportionality between ψ_{21} and I_1 is called the MUTUAL INDUCTANCE:

$$M_{21} = \frac{\Psi_{21}}{I_1} \qquad \frac{\text{Magnetic flux through circuit 2}}{\text{(due to) Current in circuit 1}}$$

From Faraday's Law, the induced emf is given by $\mathcal{E}_{21} = -M_{21}\dfrac{dI_1}{dt}$

Note:

- M_{21} depends on the shape, size, numbers of turns and relative positions of the two circuits
- $M_{21} = M_{12}$ so we need only use M as the symbol for mutual inductance
- *Ohanian* uses L for mutual inductance
- In the SI system, inductance is measured in Henrys (H)

  1 H = Inductance that produces an emf of 1 Volt for a rate of change of current of 1 A s^{-1}

  $1\ \text{H} \equiv 1\ \text{V s A}^{-1}$

- The usual unit for μ_o is H m^{-1}

Self inductance

Even a single circuit produces a magnetic field that passes through the circuit itself.

So, if I changes $\rightarrow$ $\overline{\mathbf{B}}$ changes $\rightarrow \Psi$ changes $\rightarrow$ induced emf

Definition: SELF INDUCTANCE

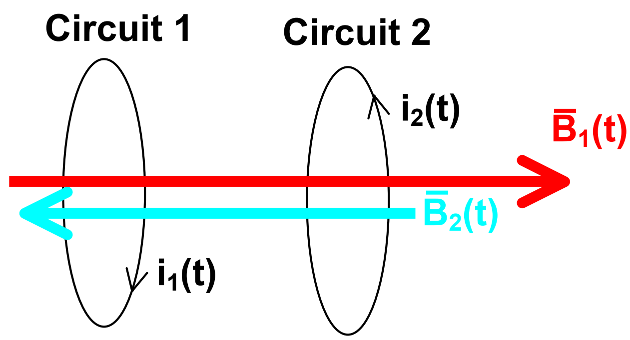
$$L = \frac{\Psi}{I} \qquad \frac{\text{Magnetic flux through a circuit}}{\text{(due to) Current in the circuit itself}}$$

Inductance and Lenz's law

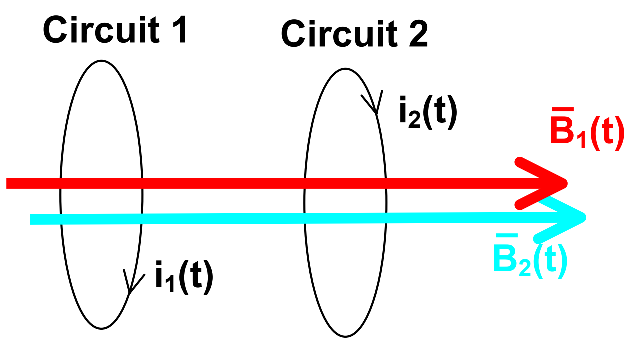
$$M \text{ or } L = \frac{\Psi}{I} \qquad \mathcal{E}_{21} = -M\frac{dI_1}{dt} \text{ or } -L\frac{dI}{dt}$$

Recall: The negative sign represents Lenz's Law: the emf causes current to flow so as to oppose the change in flux that produces it.

If ψ is $\left\{\begin{array}{l}\text{INCREASING}\\ \text{DECREASING}\end{array}\right\}$ then $\mathcal{E}$ will cause current to flow so as to create a magnetic field that $\left\{\begin{array}{l}\text{OPPOSES}\\ \text{SUPPORTS}\end{array}\right\}$ ψ.



- $i_1(t)$ increases
- ψ_{21} increases
- Induced emf in circuit 2 drives current $i_2(t)$
- $I_2(t)$ generates $\overline{\mathbf{B}}_2(t)$ which OPPOSES $\overline{\mathbf{B}}_1(t)$



- $i_1(t)$ decreases
- ψ_{21} decreases
- Induced emf in circuit 2 drives current $i_2(t)$
- $I_2(t)$ generates $\overline{\mathbf{B}}_2(t)$ which SUPPORTS $\overline{\mathbf{B}}_1(t)$

Procedure for finding inductance

1. Assume that a current I flows in (one of the) circuit(s)
2. Find the magnetic field (e.g., use Ampere's Law)
3. Find the magnetic flux linked
4. Put M or L equal to ψ/I

Examples:
1. Mutual inductance of two long coaxial solenoids
2. Self inductance of a long solenoid

See lecture notes

Energy storage in inductors

Recall:

Putting a charge on a capacitor $\Rightarrow$ Doing work $\Rightarrow$ Storing potential energy $\qquad U_{elec} = \frac{1}{2} CV^2$

Similarly:

Making current flow in an inductor $\Rightarrow$ Doing work $\Rightarrow$ Storing potential energy $\qquad U_{mag} = ?$

Because: changing current $\Rightarrow$ changing $\bar{\mathbf{B}}$ $\Rightarrow$ induced emf which opposes attempt to change current (called a "back emf")

To find the energy stored in an inductor:

Let current in the inductor i = 0 initially

Apply external emf $\Rightarrow$ current increases at rate $\dfrac{di}{dt}$

This leads to a back emf $\qquad \mathcal{E}_b = -L\dfrac{di}{dt}$

To make current flow, the required external emf is $\qquad \mathcal{E}_{ext} = L\dfrac{di}{dt}$

The externally applied power is therefore $\qquad P = \mathcal{E}_{ext} i = Li\dfrac{di}{dt}$

The work done to increase the current from 0 to I is therefore

$$U = \int_0^t P dt = \int_0^I Li di \quad \Rightarrow \quad U = \frac{1}{2} LI^2$$

Energy stored in an inductor "charged" with current I

Where is the stored energy? It is in the MAGNETIC FIELD.

Recall: For a solenoid of length a, radius R, n turns per unit length:

$L = \pi R^2 \mu_o n^2 a \qquad B = \mu_o n I$

$$U = \tfrac{1}{2} L I^2 = \frac{1}{2}\left[\pi R^2 \mu_o n^2 a\right]\left[\frac{B}{\mu_o n}\right]^2 = \frac{1}{2}\left[\frac{B^2}{\mu_o}\right]\left[\pi R^2 a\right]$$

$$\text{Therefore } U = \frac{1}{2}\left[\frac{B^2}{\mu_o}\right]\left[\text{Volume of the solenoid}\right]$$

The ENERGY DENSITY OF THE MAGNETIC FIELD is therefore

$$u = \frac{1}{2}\left[\frac{B^2}{\mu_o}\right]$$

This holds generally, for any magnetic field.