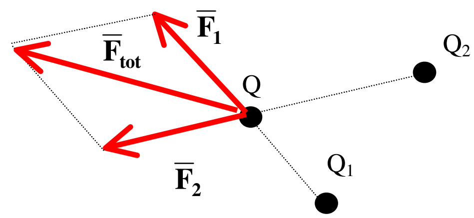
Answer 1

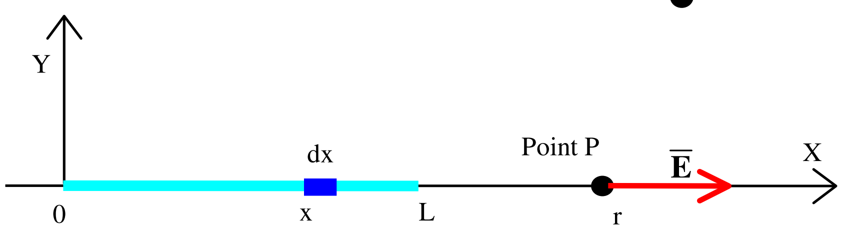
(a) Principle of Superposition: The force on a charge due to a number of other charges is given by the vector sum of all the forces on it due to the individual charges.

e.g.: Total force on Q is

$$\overline{\mathbf{F}}_{tot} = \overline{\mathbf{F}}_1 + \overline{\mathbf{F}}_2$$



(b)



Clearly, $\overline{\mathbf{E}}$ points along the +x direction (the direction in which a positive charge would tend to move if placed at the point).

To find the magnitude of $\overline{\mathbf{E}}$, consider a small element of the rod, dx, at a distance x from the origin. We can regard this as a point charge, find its contribution to the field at the point in question, and then integrate over the whole rod to find the total field.

The charge on dx is $dQ = \lambda dx$.

The contribution of dx to E at point P is $$dE = \frac{\lambda dx}{4\pi\varepsilon_0 (r-x)^2}$$

So the total field at P is $$E = \left[\frac{\lambda}{4\pi\varepsilon_0}\right]\int_0^L \frac{1}{(r-x)^2}dx$$

Let $u = r - x$ $\quad du = -dx$ $\quad x = 0,\ u = r$ $\quad x = L, u = r - L$

Therefore $$E = -\left[\frac{\lambda}{4\pi\varepsilon_0}\right]\int_r^{r-L}\frac{1}{u^2}du = -\left[\frac{\lambda}{4\pi\varepsilon_0}\right]\left[-\frac{1}{u}\right]_r^{r-L} = \left[\frac{\lambda}{4\pi\varepsilon_0}\right]\left[\frac{1}{u}\right]_r^{r-L}$$

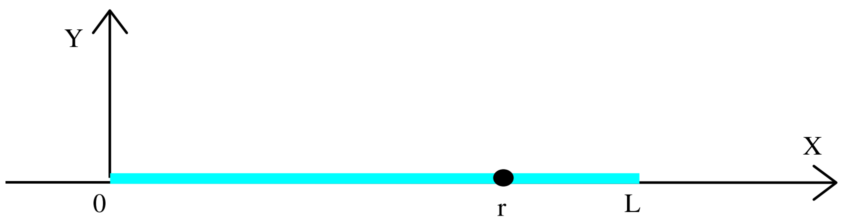
So $$E = \frac{\lambda L}{4\pi\varepsilon_0 r(r-L)}.$$

As noted above, $\overline{\mathbf{E}}$ points in the x-direction, so we write $$\overline{\mathbf{E}} = \frac{\lambda L}{4\pi\varepsilon_0 r(r-L)}\hat{\mathbf{i}}.$$

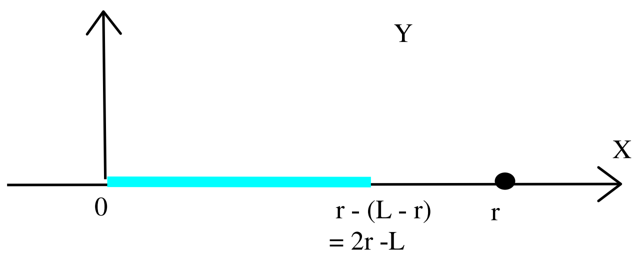
If $r >> L$ then $\overline{\mathbf{E}} \approx \dfrac{\lambda L}{4\pi\varepsilon_0 r^2}\hat{\mathbf{i}} \approx \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{i}}$,

where Q is the total charge on the rod. This is consistent with the fact that the rod appears as a point charge for r >> L, so that the electric field is given by Coulmb's Law.

(c) Consider a point inside the rod, a distance r from the origin. By symmetry, the contributions to the electric field from the sections of length L - r on either side of it will cancel out.



So this is equivalent to:



We can now use the above result with L replaced by 2r - L:

$$\overline{\mathbf{E}} = \frac{\lambda(2r - L)}{4\pi\varepsilon_0 r\left(r - (2r - L)\right)}\hat{\mathbf{i}} = \frac{\lambda(2r - L)}{4\pi\varepsilon_0 r\left(L - r\right)}\hat{\mathbf{i}} \quad .$$

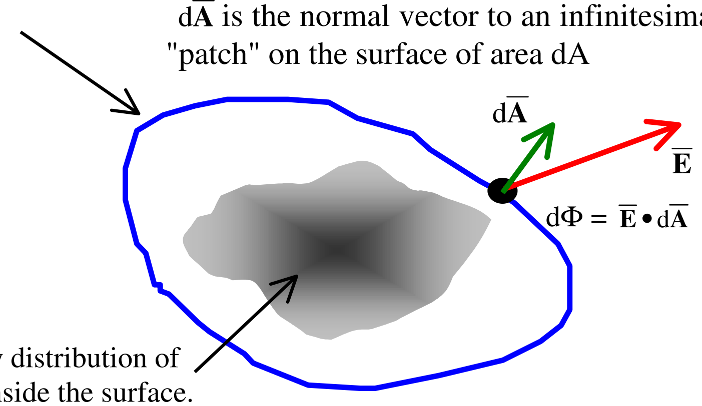
Answer 2

(a) Gauss' s Law: $\Phi = \oint \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \dfrac{Q_{enc}}{\varepsilon_o}$

where Φ = Electric flux through an arbitrary closed surface
$\overline{\mathbf{E}}$ = Electric field at the surface
$d\overline{\mathbf{A}}$ = Normal vector to an infinitesimal area, dA, on the surface
Q_{enc} = Total electric charge contained within the closed surface
ε_o = Permittivity constant
$\oint$ $\Rightarrow$ Integral over a *closed* surface

Gauss' s law states that the total electric flux, Φ, through a **closed surface** is equal to the enclosed charge divided by ε_o.

This holds regardless of the location or distribution of the charge inside.



(b)

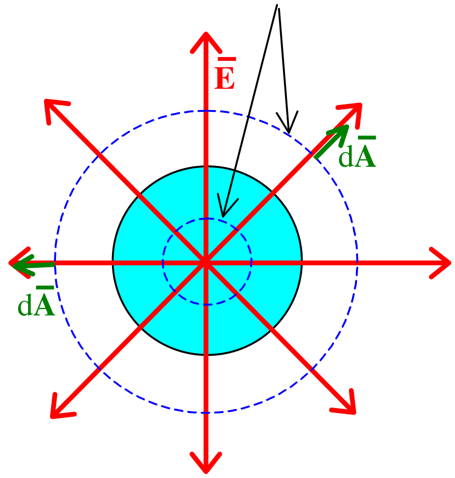
1. Field pattern: By symmetry, the electric field lines due to a spherically symmetric charge distribution must be radial.

2. Best Gaussian surface: If we choose a sphere of radius r as the Gaussian surface, the field will be perpendicular to the surface at all points, so

$\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = EdA$

Moreover, since, by symmetry, E can depend only on r and all points on the sphere have the same radius, E is constant over the spherical surface.

This applies to both of the Gaussian spheres.



3. Find Φ: $\Phi = \oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \oint EdA = E\oint dA$ = E(surface area of sphere) = $E(4\pi r^2)$

4. Find $Q_{enc.}$:

(i) r < R: Enclosed charge = (charge density)(volume of Gaussian Sphere)

$$Q_{enc} = \rho \frac{4\pi r^3}{3}$$

(i) r > R: Enclosed charge = (charge density)(Volume of charged Sphere)

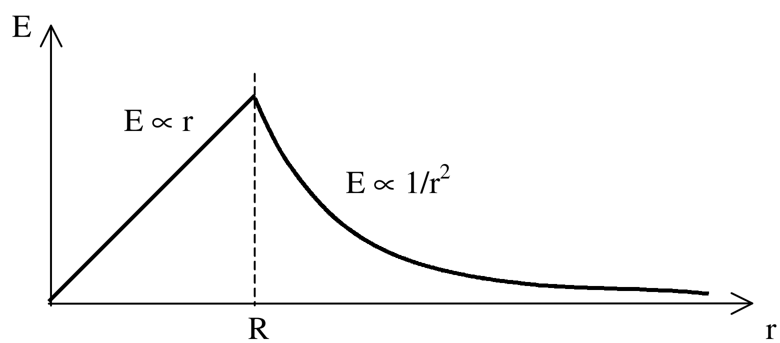
$$Q_{enc} = \rho \frac{4\pi R_1^3}{3}$$

5. Combine the results of (3) and (4) to find the electric field:

(i) r < R: $4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_o} = \rho \frac{4\pi r^3}{3\varepsilon_o} \quad \Rightarrow \quad E = \frac{\rho r}{3\varepsilon_o}$.

(ii)r > R: $4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_o} = \rho \frac{4\pi R_1^3}{3\varepsilon_o} \quad \Rightarrow \quad E = \frac{\rho R_1^3}{3\varepsilon_o r^2}$.

Sketch of E vs. r:



(c) In the region between R_1 and R_2, the field is given by

$$E = \frac{\rho r}{3\varepsilon_o} \text{ - (E due to the sphere that has been removed)}$$

$$\Rightarrow \quad E = \frac{\rho r}{3\varepsilon_o} - \frac{\rho R_2^3}{3\varepsilon_o r^2} = \frac{\rho}{3\varepsilon_o}\left[r - \frac{R_2^3}{r^2}\right].$$

For $r < R_2$, a Gaussian sphere would enclose no charge, so E = 0

For $r > R_1$, the field is $E = \frac{\rho(R_1^3 - R_2^3)}{3\varepsilon_o r^2}$.

Answer 3

(a) Relationship between electric field and potential: The potential difference between two points in space, a and b, is defined as

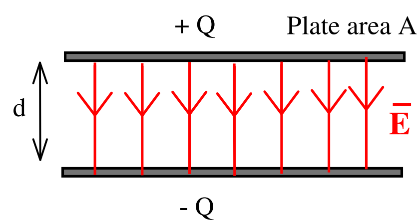
$$\Delta V = V_b - V_a = -\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}$$

or: if V(x,y,z) is the electric potential as a function of position in space, then

$$\overline{\mathbf{E}}(x,y,z) = \frac{\delta V}{\delta x}\hat{\mathbf{i}} + \frac{\delta V}{\delta y}\hat{\mathbf{j}} + \frac{\delta V}{\delta z}\hat{\mathbf{k}} .$$

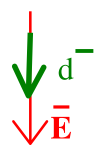
(b) Assume that one plate carries charge +Q and the other -Q. Ignoring edge effects, the electric field between the plates is uniform and given by $E = \sigma/K\varepsilon_0$.

But $\sigma = Q/A$ so $E = \dfrac{Q}{K\varepsilon_0 A}$



To find the potential difference between the plates, we integrate the electric field following any path from one plate to the other.

If we follow a field line, then $d\overline{\mathbf{L}}$ is always parallel to $\overline{\mathbf{E}}$ and $\overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = EdL$



So $$\Delta V = \int_0^d \frac{Q}{K\varepsilon_0 A} dL = \frac{Qd}{K\varepsilon_0 A} .$$ By definition, $C = \dfrac{Q}{\Delta V}$, so $C = \dfrac{K\varepsilon_0 A}{d}$.

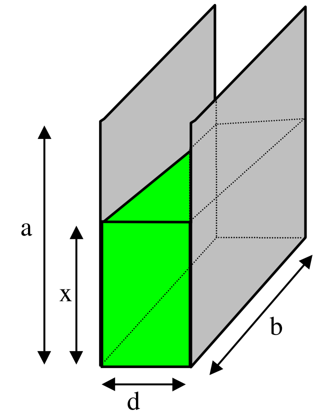
(c) The total energy of the system is $U = \dfrac{1}{2}Q\Delta V = \dfrac{1}{2}\dfrac{Q^2 d}{K\varepsilon_0 A}$ or

$$U = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\frac{K\varepsilon_o A}{d}\frac{Q^2 d^2}{K^2 \varepsilon_o{}^2 A^2} = \frac{1}{2}\frac{Q^2 d}{K\varepsilon_0 A} .$$

Regard this energy as existing in the uniform electric field between the plates, which occupies a volume of Ad.

The energy per unit volume is therefore $u = \dfrac{1}{2}\dfrac{Q^2}{K\varepsilon_o A^2}$.

But $Q/A = K\varepsilon_0 E$, so $u = \dfrac{1}{2}K\varepsilon_o E^2$.



(d) This system is equivalent to two capacitors in parallel.

For x = a, the capacitance is

$$C_{max} = \frac{4\varepsilon_0 ab}{d}.$$

For x < a, the capacitance is

$$C(x) = \frac{4\varepsilon_0 xb}{d} + \frac{\varepsilon_0 (a-x)b}{d} = \frac{\varepsilon_0 b}{d}(3x+a).$$

The ratio of the measured to the maximum value of capacitance is

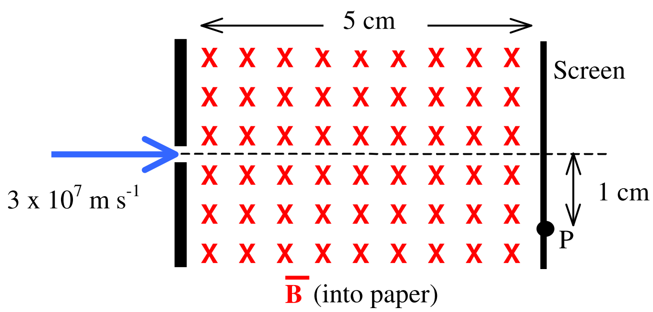
$$\frac{C(x)}{C_{max}} = R = \frac{3x+a}{4a} \qquad \Rightarrow \qquad x = \left[\frac{4R-1}{3}\right]a.$$

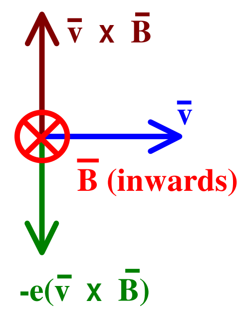
Answer 4

(a) $\overline{\mathbf{F}} = Q\left[\overline{\mathbf{E}} + (\overline{\mathbf{v}} \times \overline{\mathbf{B}})\right]$

The direction of the magnetic force is that of the cross product $\overline{\mathbf{v}} \times \overline{\mathbf{B}}$, which is perpendicular to both vectors. Since there is never any component of the magnetic force along the direction of $\overline{\mathbf{v}}$, the speed of the particle cannot be changed by the magnetic force.

(b) The force on an electron moving in the magnetic field is $\overline{\mathbf{F}} = -e(\overline{\mathbf{v}} \times \overline{\mathbf{B}})$





Now $\overline{\mathbf{v}} \perp \overline{\mathbf{B}}$ so $F = -e\,vB$

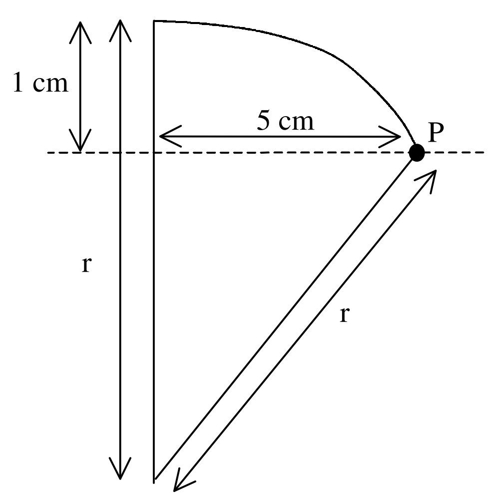
$\overline{\mathbf{F}} \perp \overline{\mathbf{v}}$ always and v never changes $\Rightarrow$ circular motion

Putting the centripetal force equal to the magnetic force, we get

$$evB = \frac{m_e v^2}{r} \quad \Rightarrow \quad r = \frac{m_e v}{eB}.$$

$$e = 1.602 \times 10^{-19} \text{ C}$$
$$m_e = 9.11 \times 10^{-31} \text{ kg} \qquad \Rightarrow \qquad r = \frac{1.71 \times 10^{-4}}{B} \text{ m.}$$
$$v = 3.0 \times 10^{7} \text{ m s}^{-1}$$

If the electrons strike the screen at point P, then



$(r - 1)^2 + 5^2 = r^2.$

So $r^2 - 2r + 1 + 25 = r^2.$

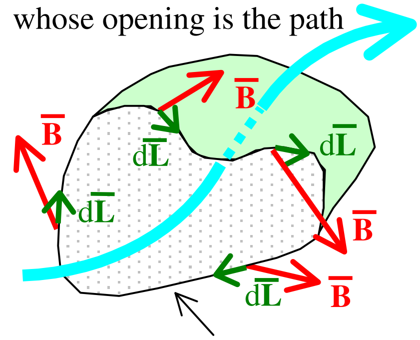
$\Rightarrow r = 13$ cm.

Therefore $0.13 = \dfrac{1.71 \times 10^{-4}}{B}$

$\Rightarrow \quad B = 1.31$ mT.

Answer 5

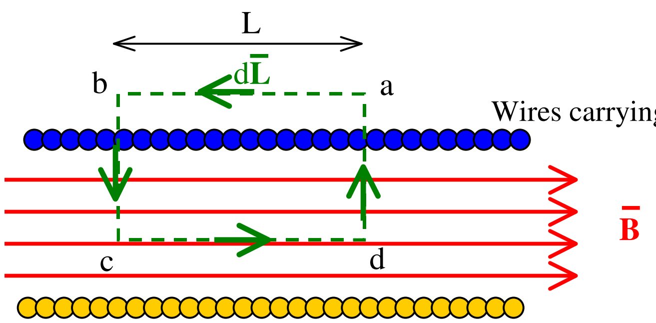
(a) $$\oint \overline{\mathbf{B}}.d\overline{\mathbf{L}} = \mu_0 I_{enc}.$$



$\overline{\mathbf{B}}$ = magnetic field
$d\overline{\mathbf{L}}$ = an infinitesimal line element vector
μ_o = permeability constant
I_{enc} = current flowing through a specified closed path
$\oint$ $\Rightarrow$ The integral is to be carried out over a *closed* path

Ampere' s law states that the line integral of the magnetic field around any closed path is equal to the total current flowing through the area defined by the path multiplied by μ_o.

(b) Consider the solenoid seen as if it were sliced down the middle:



Apply Ampere' s law to the dotted rectangular path:

$$\oint \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = \int_a^b \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} + \int_b^c \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} + \int_c^d \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} + \int_d^a \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}}$$

$$\int_a^b \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = 0 \quad \text{because } B = 0$$

$$\int_b^c \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = \int_d^a \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = 0 \text{ because } B = 0 \text{ outside the solenoid and } d\overline{\mathbf{L}} \perp \overline{\mathbf{B}} \text{ inside}$$

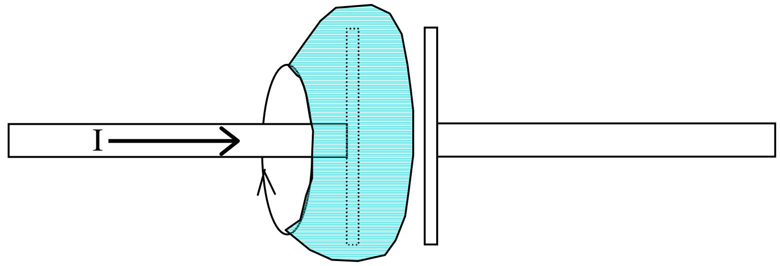
$$\int_c^d \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = \int_c^d BdL = B\int_c^d dL \qquad \text{as } d\overline{\mathbf{L}} \text{ and } \overline{\mathbf{B}} \text{ are parallel and B is constant}$$

$\int_c^d dL$ is just the length L. Therefore $\oint \overline{\mathbf{B}}\cdot d\overline{\mathbf{L}} = BL$

Enclosed current: nL wires are intercepted by the path, each carrying a current I. So $I_{enc} = nLI$.

Therefore $BL = \mu_o nLI$ so $B = \mu_o nI$.

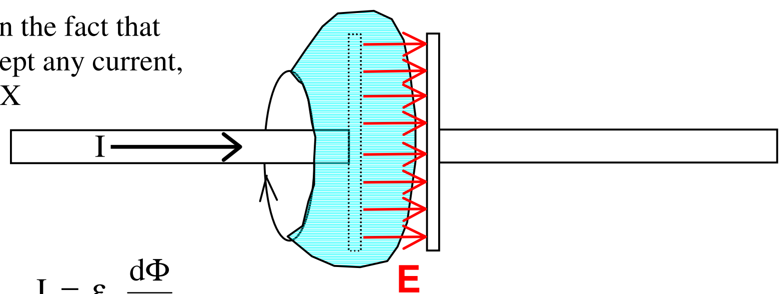
(c) Consider a parallel plate capacitor being charged up by a current I:



We know that a moving charge produces a magnetic field, so the current in the wire will generate a magnetic field. Applying Ampere' s law to the path shown, with the bag-like surface passing between the capacitor plates, we get

$$\oint \overline{\mathbf{B}}.d\overline{\mathbf{L}} = \mu_o I = 0 \Rightarrow \overline{\mathbf{B}} \text{ must be zero, which we know cannot be true.}$$

Maxwell' s modification is based on the fact that although the surface does not intercept any current, it DOES intercept ELECTRIC FLUX



How much flux is intercepted ?
Let Q = charge on capacitor

$$\Phi = \frac{Q}{\varepsilon_o} \Rightarrow \frac{d\Phi}{dt} = \frac{I}{\varepsilon_o} \Rightarrow I = \varepsilon_o \frac{d\Phi}{dt}$$

$\Rightarrow$ If we want $\oint \overline{\mathbf{B}}.d\overline{\mathbf{L}} = \mu_0 I$ as for a straight wire, we must claim that

$$\oint \overline{\mathbf{B}}.d\overline{\mathbf{L}} = \underbrace{\mu_0 I}_{\text{Term 1}} + \underbrace{\mu_0 \varepsilon_0 \frac{d\Phi}{dt}}_{\text{Term 2}}$$

Continuous wire : Term 1 = $\mu_o I$
Term 2 = 0

Capacitor : Term 1 = 0
Term 2 = $\mu_o \varepsilon_o \dfrac{d\Phi}{dt} = \mu_o I$

So the magnetic field is the same in each case

Maxwell showed that this is a general relation which holds always.

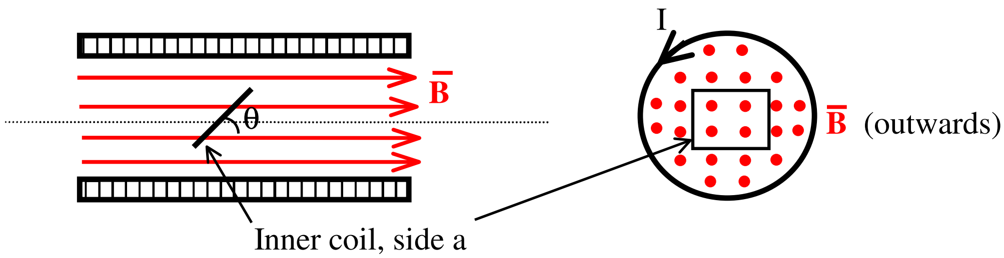
Answer 6

(a) Mutual inductance between two circuits: if a circuit carries current I_1 and a second circuit links an amount of magnetic flux Ψ_{21} due to I_1 then the mutual inductance, M, is defined by

$M = \Psi_{21}/I_1$ (Inductance = Flux/Current).

The emf induced in the second circuit is $E_2 = -M(dI_1/dt)$.

(b) Long solenoid of radius a, n turns per unit length, current I_1, $B = \mu_0 n I_1$.



SIDE VIEW **AXIAL VIEW**

(i) $$M = \frac{\Psi_{21}}{I_1} = \frac{B(\text{No. of turns of inner coil})(\text{Projected area of inner coil})}{I_1} = \frac{\mu_0 n I_1 N a^2 \sin\theta}{I_1}.$$

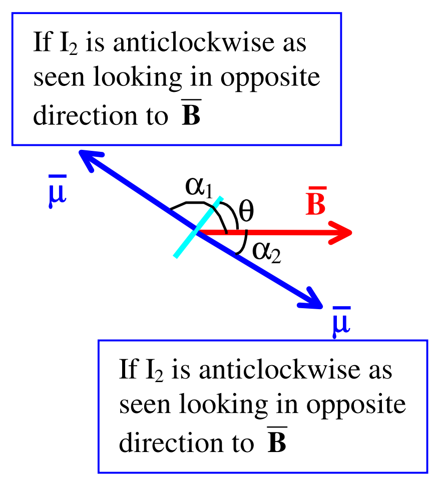
Therefore $M = \mu_0 n N a^2 \sin\theta$.

(ii) Current in inner coil = I_2.

$\Rightarrow$ Magnetic dipole moment vector of inner coil has

Magnitude: $\mu = Na^2 I_2$ (current)(area)(no. of turns)

Direction: perpendicular to the coil, the exact direction depending on whether the current is clockwise or anticlockwise.



Magnitude of torque on inner coil = magnitude of $\bar{\tau} = \bar{\mu} \times \bar{\mathbf{B}}$

$\tau = \mu B \sin\alpha_1$ or $\mu B \sin\alpha_2$ $\Rightarrow$ $\tau = \mu B \cos\theta = (Na^2 I_2)(\mu_0 n I_1)\cos\theta$.

So $\tau = \mu_0 N n a^2 I_1 I_2 \cos\theta$.

(iii) Work done in rotating the coil through 180^o starting from $\theta = 90^0$:

This corresponds to rotation between $\theta = \pi/2$ (the equilibrium position) and $\theta = 3\pi/2$. To overcome the magnetic torque, we must provide an external opposing torque $-\tau$. When the coil is at an angle θ to the x axis and is rotated by an infinitesimal angle $d\theta$, the work done is thus $-\tau d\theta$. The total work done is therefore

$$W = -\int_{\pi/2}^{3\pi/2} \mu_o n N a^2 I_1 I_2 \cos\theta \, d\theta = -\mu_o n N a^2 I_1 I_2 [\sin\theta]_{\pi/2}^{3\pi/2} = -\mu_o n N a^2 I_1 I_2 [-1 - 1] = 2\mu_o n N a^2 I_1 I_2$$

The new position is one of unstable equilibrium: the torque increases as the coil moves out of position. If the coil were displaced slightly from this position, the torque would rotate it back towards the stable equilibrium position ($\theta = 90^o$).