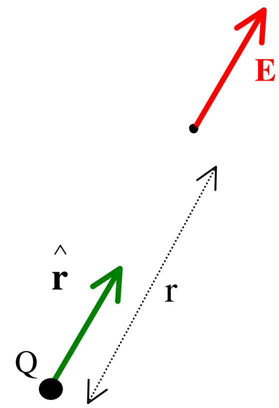
Answer 1

(a) The electric field at a point in space is defined as the electric force which a charge Q would experience at that position, divided by Q (i.e., the force per unit charge).



At distance r from a point charge,

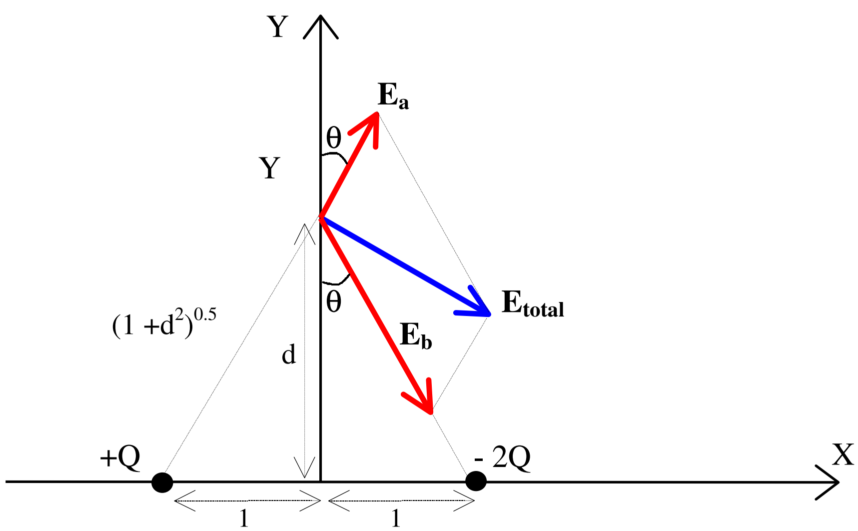
$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\hat{\mathbf{r}} \qquad \text{where}$$

$\mathbf{E}$ = electric field vector
Q = electric charge
r = distance between the charge and the point in question
ε_o = permitivity constant
$\hat{\mathbf{r}}$ = a unit vector pointing from the charge toward the point.

[5]

(b) There are two contributions to the electric field at point (0,d):

$\mathbf{F_a}$ due to +Q at (-1, 0) $\qquad$ $\mathbf{F_b}$ due to -2Q at (1, 0)



Resolving $\mathbf{F_a}$ and $\mathbf{F_b}$ into their horizontal and vertical components, we have

$$\mathbf{E}_a = E_a \sin\theta\ \hat{\mathbf{i}} + E_a \cos\theta\ \hat{\mathbf{j}},$$

and

$$\mathbf{E}_b = E_b \sin\theta\ \hat{\mathbf{i}} - E_b \cos\theta\ \hat{\mathbf{j}}.$$

From Coulomb's law, $\quad E_a = \dfrac{Q}{4\pi\varepsilon_o(1+d^2)}$ and $E_b = \dfrac{2Q}{4\pi\varepsilon_o(1+d^2)}$, and from diagram, $\cos\theta = \dfrac{d}{\left[1+d^2\right]^{1/2}}$ and $\sin\theta = \dfrac{1}{\left[1+d^2\right]^{1/2}}$.

Inserting these into the above expressions for $\mathbf{E_a}$ and $\mathbf{E_b}$, we get

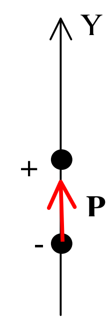
$$\mathbf{E}_a = \frac{Q}{4\pi\varepsilon_o(1+d^2)}\left[\frac{1}{(1+d^2)^{0.5}}\,\hat{\mathbf{i}} + \frac{d}{(1+d^2)^{0.5}}\,\hat{\mathbf{j}}\right], \text{ and}$$

$$\mathbf{E}_b = \frac{2Q}{4\pi\varepsilon_o(1+d^2)}\left[\frac{1}{(1+d^2)^{0.5}}\,\hat{\mathbf{i}} - \frac{d}{(1+d^2)^{0.5}}\,\hat{\mathbf{j}}\right].$$

Adding these gives $\mathbf{E}_{total} = \mathbf{E}_a + \mathbf{E}_b = \dfrac{Q}{4\pi\varepsilon_o\left(1+d^2\right)^{3/2}}\left[3\hat{\mathbf{i}} - \mathrm{d}\,\hat{\mathbf{j}}\right]$

[12]

(c) By definition, the dipole moment vector, **P**, points from the negative dipole charge towards the positive dipole charge, so here we have



$$\mathbf{P} = P\,\hat{\mathbf{j}}.$$

The torque is given by $\boldsymbol{\tau} = \mathbf{P} \times \mathbf{E}$,

where **E** is the electric field as given by the above equation for $\mathbf{E}_{tot}$ with d = 3.

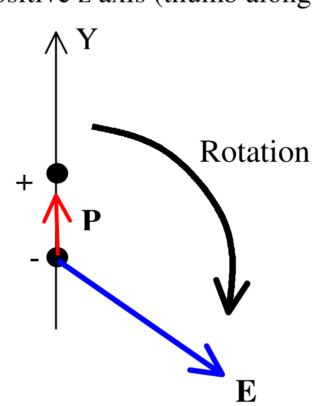
Therefore $\boldsymbol{\tau} = P\,\hat{\mathbf{j}} \times \dfrac{Q}{4\pi\varepsilon_o 10^{3/2}}\left[3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}\right]$

Now, $\hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0$, and, using the right hand rule, $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$.

Therefore $\boldsymbol{\tau} = -\dfrac{3PQ}{4\pi\varepsilon_o 10^{3/2}}\,\hat{\mathbf{k}}$.

[8]

(d) By the right hand rule, torque in the -z direction corresponds to clockwise rotation as seen looking down to the origin from a position on the positive z axis (thumb along direction
of **τ**, fingers curl in sense of rotation).

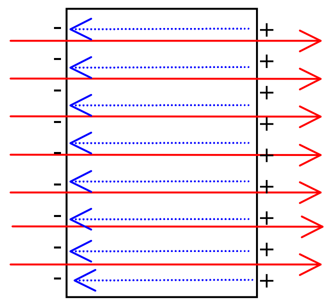


Alternative explanation: the torque will tend to align the dipole moment vector with the electric field which in this case has a positive x-component and a negative y-component.

[3]

TOTAL = 30 marks

Answer 2



(a) (i) E = 0 inside a perfect conductor:

Let a conductor be placed in a field **E** .

Charges are free to move inside the conductor and will do so under the influence of **E**

⇒ positive charge builds up on the right and negative charge on the left

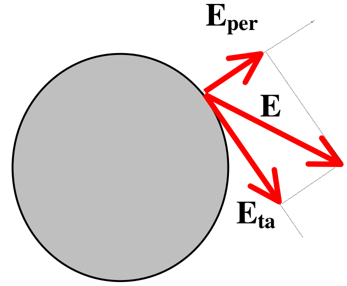
⇒ an internal field $\mathbf{E_i}$ is generated which opposes **E**. $\mathbf{E_{tot}} = \mathbf{E} + \mathbf{E_i}$.

If $E_i < E$, then E_{tot} is in the same direction as E and the charges continue to build up causing E_i to increase.

If $E_i > E$, then E_{tot} is in the opposite direction as E and some charges move back again causing E_i to decrease.

Therefore, in equilibrium, $\mathbf{E_i} = -\mathbf{E}$, so $\mathbf{E_{tot}} = 0$.

[4]



(ii) At the surface of the conductor, the **E** is perpendicular to the surface:

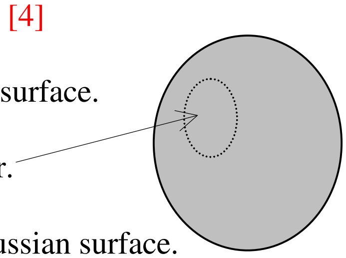
Assume that for some reason **E** is not perpendicular to the surface.

Let $\mathbf{E_{perp}}$ and $\mathbf{E_{tan}}$ be the components of **E** perpendicular and tangential to the surface, respectively.

If E_{tan} is not zero then charges will move along the surface, setting up an opposing field (as argued above).

⇒ An equilibrium is established in which $\mathbf{E_{tan}}$ is exactly cancelled by the opposing field due to the separation of charges it creates.

⇒ $\mathbf{E_{tot}}$ has only the $\mathbf{E_{perp}}$ component. [4]



(iii) All excess charge on the conductor is located at its surface.

Consider any Gaussian surface inside the conductor.

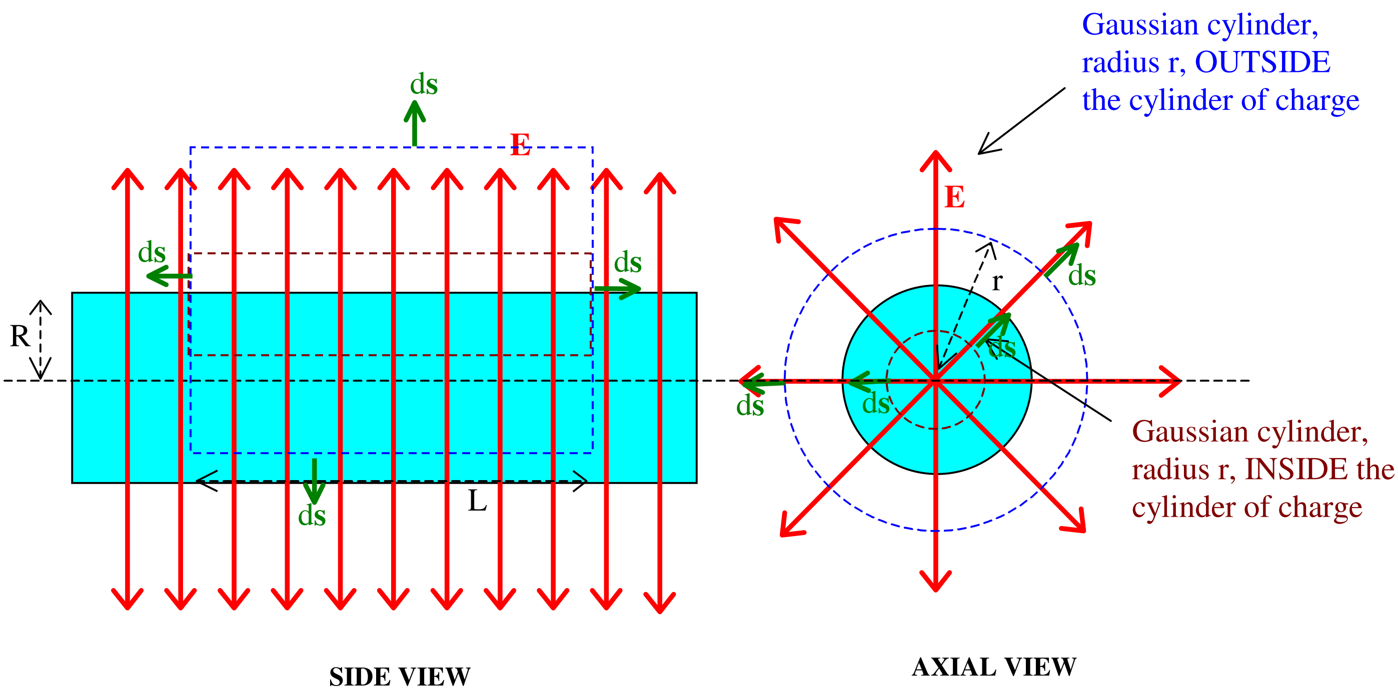
From (i), we have $\mathbf{E} = 0$ at every point on this Gaussian surface.

⇒ By Gauss's Law, the total electric flux, Φ, through the Gaussian surface, is

$$\Phi = \oint \mathbf{E}.d\mathbf{s} = 0 \Rightarrow Q_{enclosed} = 0.$$

[4]

b) Step 1 Field pattern: By symmetry, the electric field lines due to a cylindrically symmetric charge distribution must be radial, as shown on the side and axial views



Step 2 Best Gaussian surface: Choose a co-axial cylinder of radius r and length L as the Gaussian surface. We need two of these, one with r < R and one with r > R.

- On the curved parts, **E** is perpendicular to the surface at all points. **E** and the normal vector d**s** are therefore parallel, so **E.ds** = Eds. Also, by symmetry, E is constant since all points are at the same distance from the axis.
- On the flat ends, **E** is perpendicular to d**s** at all points so **E.ds** = 0.

Step 3 Find Φ: We can split the surface integral into three parts

$$\Phi = \oint \mathbf{E.ds} = \int_{Top} \mathbf{E.ds} + \int_{Bottom} \mathbf{E.ds} + \int_{Side} \mathbf{E.ds} = 0 + 0 + \int_{Side} \mathbf{E.ds}$$

$$\Rightarrow \quad \Phi = \int_{Side} E ds = E \int_{Side} ds = E(\text{surface area of curved part}) = E(2\pi r L)$$

Step 4 Find $Q_{enc.}$:

For $r < R$: The Gaussian cylinder is full of charge, and the charge enclosed is given by the volume times the charge density:

$$Q_{enc} \;=\; \pi r^2 L\rho \,.$$

For $r > R$: The Gaussian cylinder is full of charge only out to a radius R, and the charge enclosed is given by the charge density times the volume of a length L of the charged cylinder:

$$Q_{enc} \;=\; \pi R^2 L\rho \,.$$

Step 5 Combine the results of (3) and (4) to find the electric field:

Put $\Phi = Q_{enc}/\varepsilon_o$ and rearrange to get an expression for E:

For $r < R$: $E = \dfrac{\rho r}{2\varepsilon_o}$ For $r < R$: $E = \dfrac{\rho R^2}{2\varepsilon_o r}$

[14]

(c) V = 0 on the axis of the cylinder. The potential at the surface is the integral of the electric field between r = 0 and r = R. This is most easily evaluated by integrating along a radial path, parallel to a field line. Ignoring the sign of V for the moment,

$$|V| = \left|\int_0^R \mathbf{E}.d\mathbf{L}\right| = \left|\int_0^R \frac{\rho r}{2\varepsilon_o}dr\right| = \left|\frac{\rho R^2}{4\varepsilon_o}\right|$$

To decide if V is positive or negative: Clearly a positive charge, once disturbed from the unstable equilibrium position on the axis, would be pushed radially outwards by the electric field. It would therefore gain kinetic energy and lose potential energy. V is therefore negative.

[4]

TOTAL = 30 marks

Answer 3

(a) The electric field vector is -(the gradient of the potential):

$$\mathrm{E} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$ [3]

(b) $V(x,y,z) = (x^2z + 2y^2z - 3z^3)$ V

(i) Taking partial derivatives,

$$E = -(2xz)\,\hat{\mathbf{i}} - (4yz)\,\hat{\mathbf{j}} - (x^2 + 2y^2 - 9z^2)\,\hat{\mathbf{k}} \qquad \text{V m}^{-1}$$

[5]

(ii) Energy change in moving -2e from a point (2,1,2) to the origin:

At (2,1,2), $V = (2^2)(2) + 2(1^2)(2) - 3(2^3) = 8 + 4 - 24 = -12$ V

At (0,0,0), $V = 0$

The magnitude of the work done is $|W| = |Q(\Delta V)| = (2e)(12) = 24$ eV

or $(24)\text{x}(1.60 \text{ x } 10^{-19}) = 3.84 \text{ x } 10^{-18}$ J.

(2,1,2) is at a lower potential than (0,0,0) so positive charge would be pushed by the field from (0,0,0) to (2,1,2). But negative charge would need to be pulled, and external work would therefore need to be done on the ion.

[5]

(iii) Electron released from rest at coordinates (0,0,1).

Initial acceleration:

The electric field at (0,0,1) is $\mathbf{E} = 9(1)^2\,\hat{\mathbf{k}} = 9\,\hat{\mathbf{k}}$ V m^{-1}

The electron will be accelerated in the -z direction (opposite to the field direction). The magnitude of the acceleration of electron is

$$a = \frac{eE}{m_e} = \frac{(1.60\,x\,10^{-19})(9)}{9.11\,x\,10^{-31}} = 1.58 \text{ x } 10^{12} \text{ m s}^{-2}$$

[5]

(iv) Equipotential line $\Rightarrow$ V = Constant
Plane parallel to x-y plane $\Rightarrow$ z = Constant

Therefore, from the equation for V, $x^2 + 2y^2 = \dfrac{V + 3z^3}{z}$

The right hand side of this equation is a constant, so the equation corresponds to an ellipse .

[5]

(v) The dipole moment vector, **P**, has magnitude = (Charge)(Separation) and direction from the negative towards the positive charge:

$\mathbf{P} = (2 \text{ x } 10^{-6})(1 \text{ x } 10^{-6})\ \hat{\mathbf{k}} = (2 \text{ x } 10^{-12})\ \hat{\mathbf{k}}$ C m

$$\mathbf{E} = -2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}} \quad \text{V m}^{-1}$$

$$\text{Torque } \boldsymbol{\tau} = \mathbf{P} \times \mathbf{E} \quad \Rightarrow \quad \boldsymbol{\tau} = \left[(2\,x10^{-12})\hat{\mathbf{k}}\right] \times \left[-2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}\right]$$

$$\Rightarrow \quad \boldsymbol{\tau} = -(4\,x10^{-12})\left[\hat{\mathbf{k}} \times \hat{\mathbf{i}}\right] - (8\,x10^{-12})\left[\hat{\mathbf{k}} \times \hat{\mathbf{j}}\right] + (1.2\,x10^{-11})\left[\hat{\mathbf{k}} \times \hat{\mathbf{k}}\right]$$

$$\Rightarrow \quad \boldsymbol{\tau} = -(4\,x10^{-12})\left[\hat{\mathbf{j}}\right] - (8\,x10^{-12})\left[-\hat{\mathbf{i}}\right] + 0$$

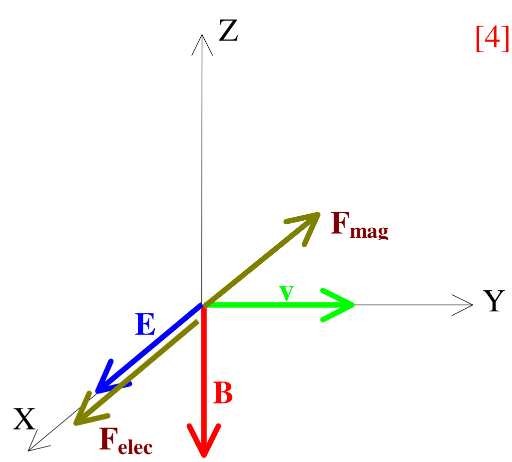
$$\Rightarrow \quad \boldsymbol{\tau} = (4\,x10^{-12})\left[2\hat{\mathbf{i}} - \hat{\mathbf{j}}\right]$$

[7]

TOTAL MARKS = 30

Answer 4

(a) $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \text{ x } \mathbf{B})]$ [4]



(b) **(i)** $\mathbf{v} = v\,\hat{\mathbf{j}} \qquad \mathbf{B} = -3\hat{\mathbf{k}}$

$\mathbf{E} = 6 \text{ x } 10^7\ \hat{\mathbf{i}}$

$\mathbf{F}_{elec} = Q\mathbf{E} = (6 \text{ x } 10^7)Q\ \hat{\mathbf{i}}$

$\mathbf{F}_{mag} = Q(\mathbf{v} \text{ x } \mathbf{B}) = (-3)v\,(\hat{\mathbf{j}} \text{ x } \hat{\mathbf{k}})$

$\hat{\mathbf{j}} \text{ x } \hat{\mathbf{k}} = \hat{\mathbf{i}}$ (by the right-hand rule) $\Rightarrow \quad \mathbf{F}_{mag} = -3Qv\ \hat{\mathbf{i}}$ [8]

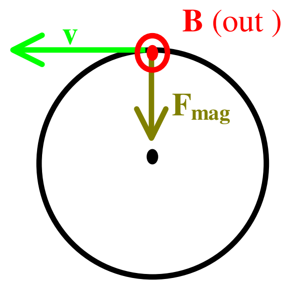
(ii) Particle is undeflected $\Rightarrow \mathbf{F}_{mag} = -\,\mathbf{F}_{elec}$

Therefore $QE = QvB \quad \Rightarrow \quad v = E/B = (6 \text{ x } 10^7)/3 = 2 \text{ x } 10^7 \text{ m s}^{-1}$ [5]

(c) If $E = 0$ then $\mathbf{F} = Q(\mathbf{v} \text{ x } \mathbf{B})$

F is always perpendicular to **v** $\Rightarrow$ v never changes - only the direction changes

F is always perpendicular to **B**



$\Rightarrow$ the force is always in the x-y plane

$\Rightarrow$ **v** remains in the x-y plane

$\Rightarrow |\mathbf{v} \times \mathbf{B}| = vB = \text{constant}$

$\Rightarrow F_{mag} = QvB$, a constant

Force of constant magnitude directed perpendicular to velocity $\Rightarrow$ motion in a circle. [8]

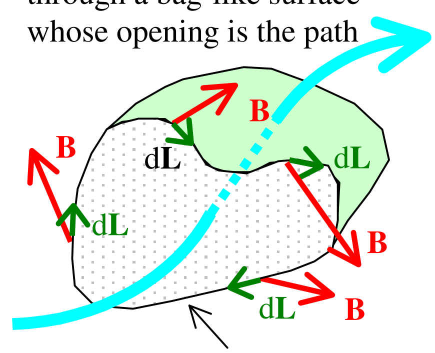
(d) Equate the magnetic force with the force needed to keep a particle moving in a circle:

$$QvB = \frac{mv^2}{r} \qquad \Rightarrow \qquad \frac{Q}{m} = \frac{v}{rB} = \frac{6 \, x \, 10^6}{(15 \, x \, 10^{-3})(3)} = 1.33 \text{ x } 10^8 \text{ C kg}^{-1}$$

[5]

TOTAL MARKS = 30

Answer 5



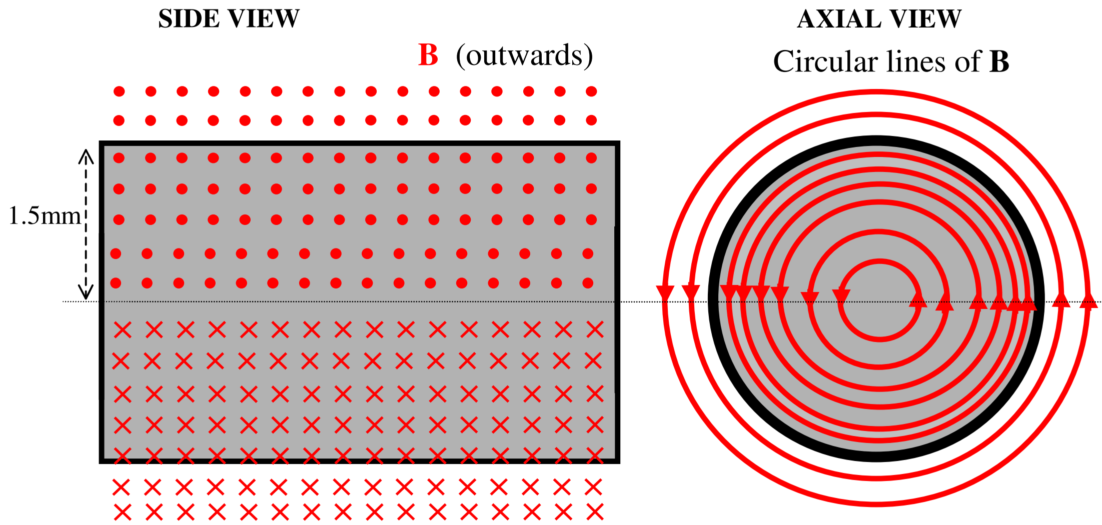
(a) $\oint \mathbf{B}.d\mathbf{L} = \mu_o I_{enclosed}$.

B = Magnetic field
d**L** = an infinitesimal line element vector
μ_o = Permeability constant
$I_{enclosed}$ = Current flowing through a specified closed path
$\oint$ $\Rightarrow$ The integral is to be carried out over a *closed* path

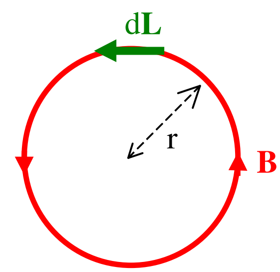
Ampere' s law states that the line integral of the magnetic field around any closed path is equal to the total current flowing through the area defined by the path divided by μ_o.

[5]

(b) By symmetry, the magnetic field lines will form circular loops around the axis.

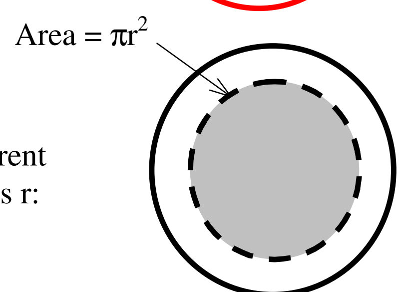


Consider any circular path of radius r centred on the axis and following a field line :



Apply Ampere's Law: **B** and d**L** are always parallel, and the magnitude of the field, B, is constant for the whole path.

So, $$\oint \mathbf{B}.d\mathbf{l} = \oint B dl = B\oint dl = B(2\pi r)$$



Enclosed current:

For $r < 1.5$ mm: The enclosed current is given by the current density multiplied by the area of the cylinder within radius r:

$$I_{enc} = 20\left[\frac{\pi r^2}{\pi(1.5x10^{-3})^2}\right] = 8.8\text{ x}10^6 r^2 \qquad \text{Amps.}$$

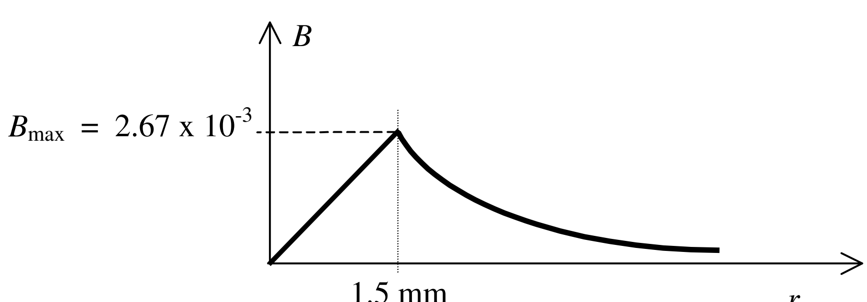
For r > 1.5 mm, the enclosed current is the total current flowing through the cylinder:

$$I_{enc} = 20 \text{ Amps.}$$

So, for $r < 1.5$ mm: $B(2\pi r) = \mu_o(8.8\text{x}10^6)r^2 = 4\pi\text{x}0.88\, r^2 \Rightarrow B = 1.76r$ Tesla (r in metres)

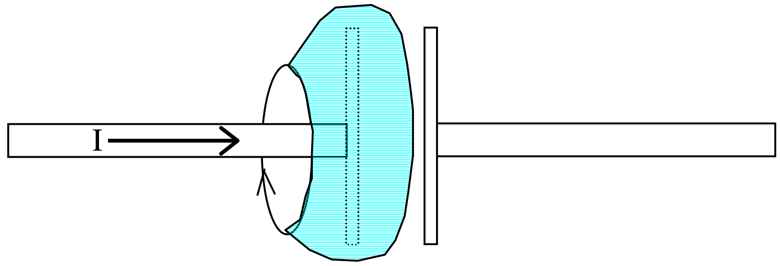
And for $r > 1.5$ mm: $B(2\pi r) = 20\mu_o \Rightarrow B = \dfrac{4 \text{ x } 10^{-6}}{r}$ Tesla (r in metres)

Sketch of B **vs.** r



[15]

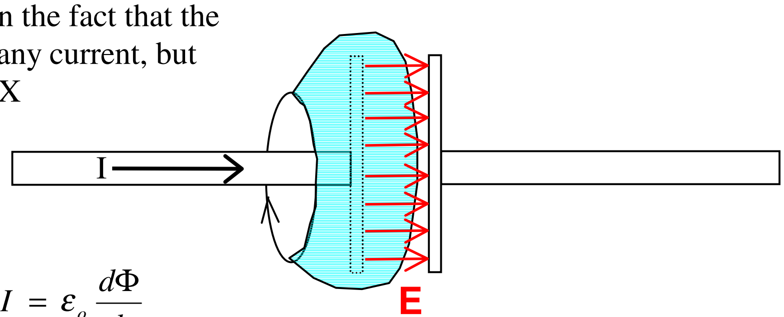
(c) Consider a parallel plate capacitor being charged up by a current I:



We know that a moving charge produces a magnetic field, so the current in the wire will generate a magnetic field, **B**. Applying Ampere' s law to the path shown, with the bag-like surface passing between the capacitor plates, we get

$$\oint \mathbf{B}.d\mathbf{L} \;=\; \mu_o I_{enc} \;=\; 0 \qquad \Rightarrow \mathbf{B} \text{ must be zero, which we know cannot be true.}$$

Maxwell' s modification is based on the fact that the although surface does not intercept any current, but it DOES intercept ELECTRIC FLUX



How much flux is intercepted ?
Let Q = charge on capacitor

$$\Phi = \frac{Q}{\varepsilon_o} \quad \Rightarrow \quad \frac{d\Phi}{dt} = \frac{I}{\varepsilon_o} \quad \Rightarrow \quad I = \varepsilon_o \frac{d\Phi}{dt}$$

$\Rightarrow$ If we want $\oint \mathbf{B}.d\mathbf{L} = \mu_o I$ as for a straight wire, we must claim that

$$\oint \mathbf{B}.d\mathbf{L} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi}{dt} \qquad [10]$$

Term 1 Term 2

Continuous wire : Term 1 = $\mu_o I$
Term 2 = 0

Capacitor : Term 1 = 0
Term 2 = $\mu_o \varepsilon_o \frac{d\Phi}{dt} = \mu_o I$

So the magnetic field is the same in each case

Maxwell showed that this is a general relation which holds always.

TOTAL MARKS = 30

Answer 6

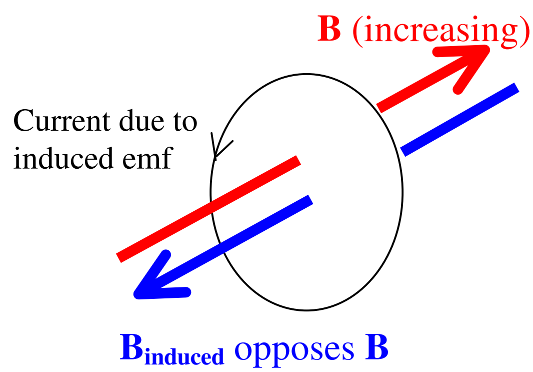
(a) If a circuit carries current i, and the amount of magnetic flux linked by the circuit (due to its own magnetic field) is Ψ then the self inductance is

$L = \Psi/i$ (Inductance = Flux/Current)

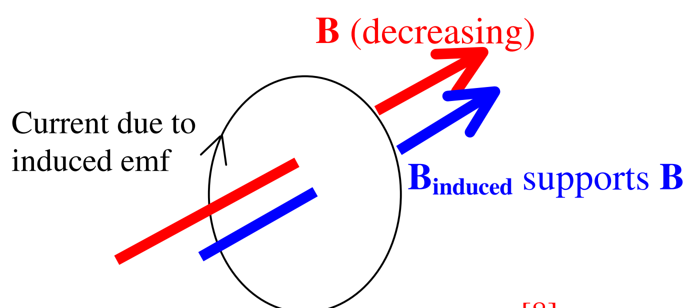
The induced emf is $\mathcal{E} = -L(di/dt)$.

Lenz's Law: The induced emf is always in such a direction as to oppose the change in magnetic flux that causes it. This is represented by the negative sign in the above equation for the emf.

If ψ is **increasing** then $\mathcal{E}$ causes current to flow so as to generate a magnetic field which **opposes** ψ.



If ψ is **decreasing** then $\mathcal{E}$ causes current to flow so as to generate a magnetic field which **supports** ψ.



[8]

(b) Consider an inductance L, and let the current, i, be increased from zero to I in time t.

The induced emf, which opposes the changing current is $\mathcal{E} = -L(di/dt)$.

In order to increase i, an opposing external voltage, v, infinitesimally greater than $\mathcal{E}$, must be applied:

$$v = L\frac{di}{dt}$$

The applied power is $P = vi = Li\dfrac{di}{dt} \quad \Rightarrow \quad Pdt = Lidi$

The total energy delivered is $U = \int_0^t Pdt = \int_0^t Lidi = L\int_0^I idi = L\dfrac{I^2}{2}$. [7]

The energy is stored in the magnetic field created by the current flowing in the coil.

[2]

(c) $L = 4\text{x}10^{-3}$ H $\quad E = 3\cos(100\pi t)$ V

(i) Current, $i(t)$: $\dfrac{di}{dt} = -\dfrac{E}{L} = -\dfrac{1}{4}(10^3)(3\cos(100\pi t))$

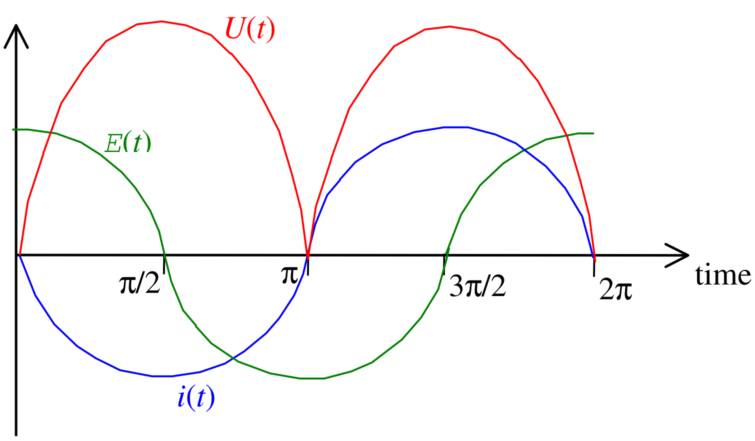
$\Rightarrow \quad i(t) = -(3/4)\text{ x }(10^3)(100\pi)^{-1}\sin(100\pi t)$

$i(t) = -2.39\sin(100\pi t)$ [4]

(ii) Magnetic flux, $\Psi(t)$: $\Psi(t) = Li(t) = -0.0096\sin(100\pi t)$ [3]

(iii) Energy stored, $U(t)$: $U(t) = 0.5Li^2 = (0.5)(4\text{x}10^{-3})(5.71)^2\sin^2(100\pi t)$

$U(t) = (0.065)\sin^2(100\pi t)$ [3]



[3]

TOTAL MARKS =30