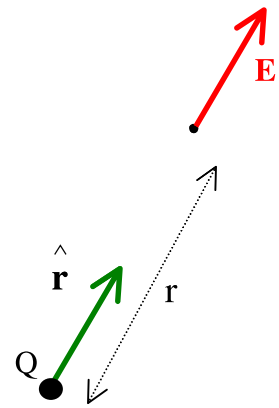
Answer 1

(a) The electric field at a point in space is defined as the electric force which a charge Q would experience at that position, divided by Q (i.e., the force per unit charge).

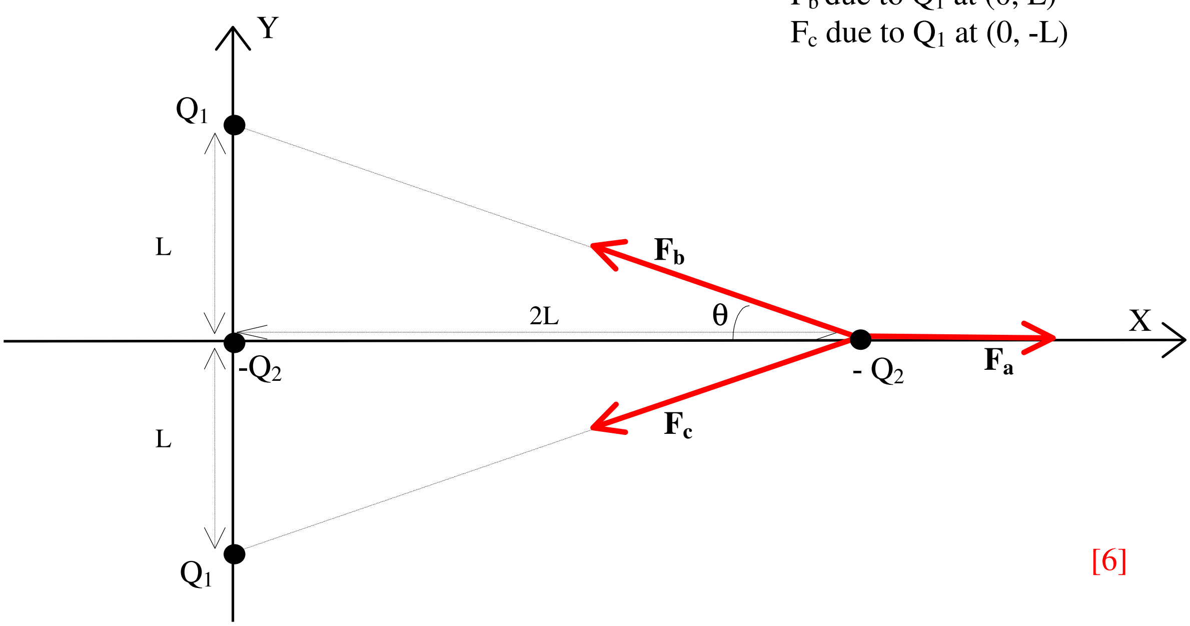


At distance r from a point charge,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\hat{\mathbf{r}} \quad \text{where}$$

$\mathbf{E}$ = electric field vector
Q = electric charge
r = distance between the charge and the point in question
ε_o = permittivity constant
$\hat{\mathbf{r}}$ = a unit vector pointing from the charge toward the point. [6]

(b) **(i)** Three forces act on the $-Q_2$ charge at (2L,0): F_a due to $-Q_2$ at (0, 0)
F_b due to Q_1 at (0, L)
F_c due to Q_1 at (0, -L)



[6]

(ii) **Resultant** $\mathbf{F_a} + \mathbf{F_b} + \mathbf{F_c} = 0$

From the diagram, this implies that $F_b\cos\theta + F_c\cos\theta = F_a$

$$F_a = \frac{(-Q_2)^2}{4\pi\varepsilon_o(2L)^2} = \frac{Q_2^{\,2}}{16\pi\varepsilon_o L^2}$$

$$F_b = F_c = \frac{Q_1Q_2}{4\pi\varepsilon_o\left[L^2 + (2L)^2\right]} = \frac{Q_1Q_2}{20\pi\varepsilon_o L^2}$$

$$\cos\theta = \frac{2L}{\left[L^2 + (2L)^2\right]^{1/2}} = \frac{2}{\sqrt{5}}$$

Therefore $$\frac{2Q_1Q_2}{20\pi\varepsilon_o L^2}\frac{2}{\sqrt{5}} = \frac{{Q_2}^2}{16\pi\varepsilon_o L^2} \quad \Rightarrow \quad Q_1 = \frac{5\sqrt{5}Q_2}{16} = 0.7Q_2$$

[8]

(iii) The electric fields due to the two Q_1 charges cancel out. Therefore electric field $\mathbf{E_d}$, at the origin is just that due to the $-Q_2$ charge at (2L,0):-

$$\mathbf{E_d} = \frac{Q_2}{16\pi\varepsilon_o L^2}\hat{\mathbf{i}} \ .$$

By definition, the dipole moment vector, **P**, points from the negative dipole charge towards the positive dipole charge, so here we have

$$\mathbf{P} = P\hat{\mathbf{j}}.$$

The torque is given by $\boldsymbol{\tau} = \mathbf{P} \times \mathbf{E}$,

where **E** is the electric field $\mathbf{E_d}$

Therefore $$\boldsymbol{\tau} = P\hat{\mathbf{j}} \times \left(\frac{Q_2}{16\pi\varepsilon_o L^2}\hat{\mathbf{i}}\right)$$

Now $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$,

Therefore $$\boldsymbol{\tau} = -\frac{PQ_2}{16\pi\varepsilon_o L^2}\hat{\mathbf{k}}$$

[10]

Answer 2

(a) Gauss's Law: $$\Phi = \oint \overline{\mathbf{E}}.\mathrm{d}\overline{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_o}$$

where Φ = Electric flux through an arbitrary closed surface
$\overline{\mathbf{E}}$ = Electric field at the surface
$\mathrm{d}\overline{\mathbf{A}}$ = Normal vector to an infinitesimal area, dA, on the surface

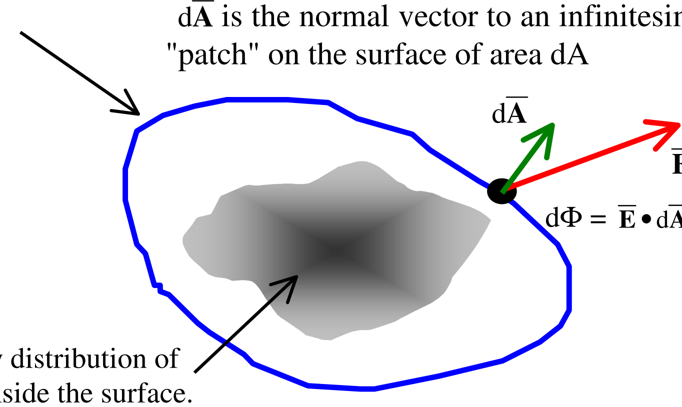
Q_{enc} = Total electric charge contained within the closed surface

ε_o = Permittivity constant

$\oint$ $\Rightarrow$ Integral over a *closed* surface

Gauss's law states that the total electric flux, Φ, through a **closed surface** is equal to the enclosed charge divided by ε_o.

This holds regardless of the location or distribution of the charge inside.



[8]

(b)

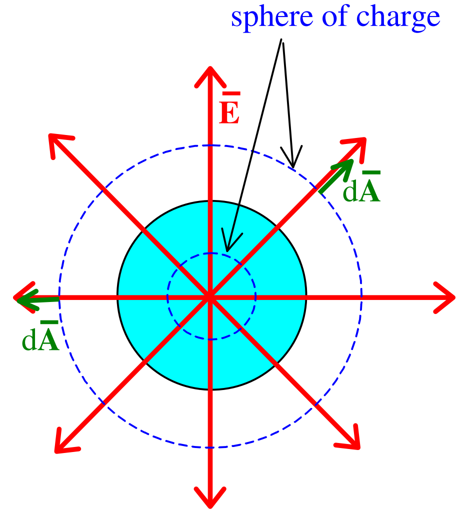
1. Field pattern: By symmetry, the electric field lines due to a spherically symmetric charge distribution must be radial.

2. Best Gaussian surface: If we choose a sphere of radius r as the Gaussian surface, the field will be perpendicular to the surface at all points, so

$$\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = EdA$$

Moreover, since, by symmetry, E can depend only on r and all points on the sphere have the same radius, E is constant over the spherical surface.

This applies to both of the Gaussian spheres.



3. Find Φ: $\Phi = \oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \oint EdA = E\oint dA = E(\text{surface area of sphere}) = E(4\pi r^2)$

4. Find $Q_{enc.}$:

(i) $r < R$: Enclosed charge = (charge density)(volume of Gaussian Sphere)

$$Q_{enc} = \rho \frac{4\pi r^3}{3}$$

(i) r > R: Enclosed charge = (charge density)(Volume of charged Sphere)

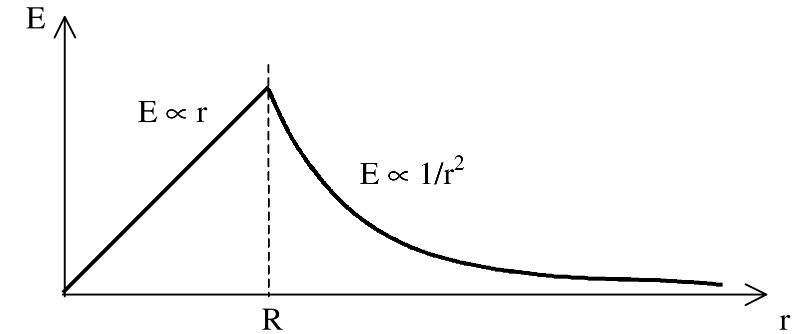
$$Q_{enc} = \rho \frac{4\pi R_1^{\,3}}{3}$$

5. Combine the results of (3) and (4) to find the electric field:

(i) $r < R$: $4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_o} = \rho \frac{4\pi r^3}{3\varepsilon_o} \Rightarrow E = \frac{\rho r}{3\varepsilon_o}.$

(ii) $r > R$: $4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_o} = \rho \frac{4\pi R_1^{\,3}}{3\varepsilon_o} \Rightarrow E = \frac{\rho R_1^{\,3}}{3\varepsilon_o r^2}.$

Sketch of E vs. r:



[13]

(c)

Magnitude of force acting on a charge of 0.5C at some point $r < R_1$ inside the sphere is

$$F = 0.5 \times E = \frac{\rho r}{6\varepsilon_o}$$

and in a direction radially outward from the centre of the sphere (assuming the latter contains positive charge.)

Hence work done in taking the charge from $r = R_1$ to $r = 0$ is postiive (since we have to work against the electric field inside the sphere) and is given by

$$-\int_{R_1}^{0} F dr = -\int_{R_1}^{0} \frac{\rho r}{6\varepsilon_0} = \frac{\rho R_1^{\,2}}{12\varepsilon_0}$$

[9]

Answer 3

(a) Relationship between electric field and potential: The potential difference between two points in space, a and b, is defined as

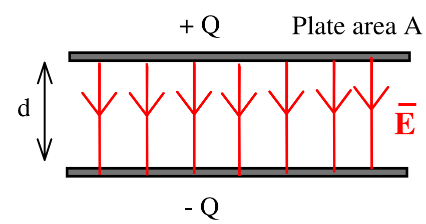
$$\Delta V = V_b - V_a = -\int_a^b \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}}$$

or: if V(x,y,z) is the electric potential as a function of position in space, then

$$\overline{\mathbf{E}}(x,y,z) = \frac{\delta V}{\delta x}\hat{\mathbf{i}} + \frac{\delta V}{\delta y}\hat{\mathbf{j}} + \frac{\delta V}{\delta z}\hat{\mathbf{k}}\,.$$ **[5]**

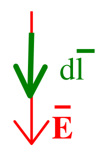
(b) Assume that one plate carries charge +Q and the other -Q. Ignoring edge effects, the electric field between the plates is uniform and given by $E = \sigma/K\varepsilon_0$.

But $\sigma = Q/A$ so $E = \dfrac{Q}{K\varepsilon_0 A}$



To find the potential difference between the plates, we integrate the electric field following any path from one plate to the other.

If we follow a field line, then $d\overline{\mathbf{L}}$ is always parallel to $\overline{\mathbf{E}}$ and $\overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = EdL$



So $\Delta V = \displaystyle\int_0^d \frac{Q}{K\varepsilon_0 A} dL = \frac{Qd}{K\varepsilon_0 A}$. By definition, $C = \dfrac{Q}{\Delta V}$, so $C = \dfrac{K\varepsilon_0 A}{d}$.

[10]

(c) The total energy of the system is $U = \dfrac{1}{2}Q\Delta V = \dfrac{1}{2}\dfrac{Q^2 d}{K\varepsilon_0 A}$ or

$$U = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\frac{K\varepsilon_o A}{d}\frac{Q^2 d^2}{K^2 \varepsilon_o^{\,2} A^2} = \frac{1}{2}\frac{Q^2 d}{K\varepsilon_0 A}\,.$$

Regard this energy as existing in the uniform electric field between the plates, which occupies a volume of Ad.

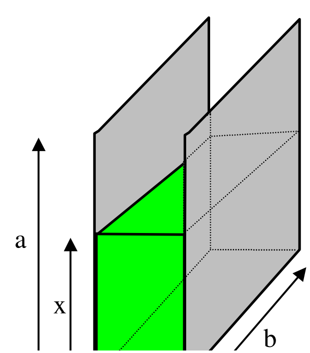
The energy per unit volume is therefore $u = \dfrac{1}{2}\dfrac{Q^2}{K\varepsilon_o A^2}$.

But $Q/A = K\varepsilon_0 E$, so $u = \dfrac{1}{2}K\varepsilon_o E^2$. **[8]**

(d) This system is equivalent to two capacitors in parallel.

So use $C(x) = C_1 + C_2$

.



$$C_1 = \frac{4\varepsilon_0 xb}{d} \quad , \quad C_2 = \frac{\varepsilon_0 (a-x)b}{d}$$

$$C(x) = \frac{4\varepsilon_0 xb}{d} + \frac{\varepsilon_0 (a-x)b}{d} = \frac{\varepsilon_0 b}{d}(3x+a)\,.$$

[7]

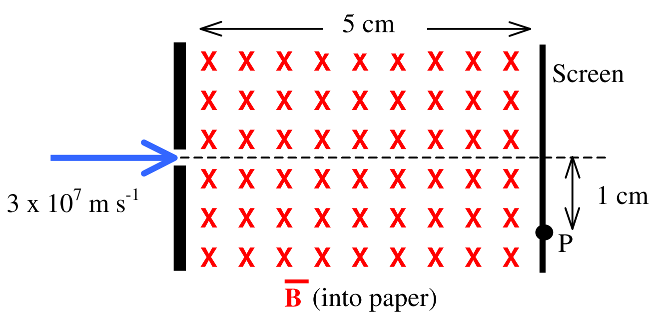
Answer 4

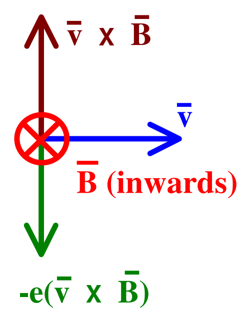
(a) $\overline{\mathbf{F}} = Q\left[\overline{\mathbf{E}} + (\overline{\mathbf{v}} \times \overline{\mathbf{B}})\right]$

The direction of the magnetic force is that of the cross product $\overline{\mathbf{v}} \times \overline{\mathbf{B}}$, which is perpendicular to both vectors. Since there is never any component of the magnetic force along the direction of $\overline{\mathbf{v}}$, the speed of the particle cannot be changed by the magnetic force.

[8]

(b) The force on an electron moving in the magnetic field is $\overline{\mathbf{F}} = -e(\overline{\mathbf{v}} \times \overline{\mathbf{B}})$





Now $\overline{\mathbf{v}} \perp \overline{\mathbf{B}}$ so $F = -e\,vB$

$\overline{\mathbf{F}} \perp \overline{\mathbf{v}}$ always and v never changes $\Rightarrow$ circular motion

Putting the centripetal force equal to the magnetic force, we get

$$evB = \frac{m_e v^2}{r} \quad \Rightarrow \quad r = \frac{m_e v}{eB}\,.$$

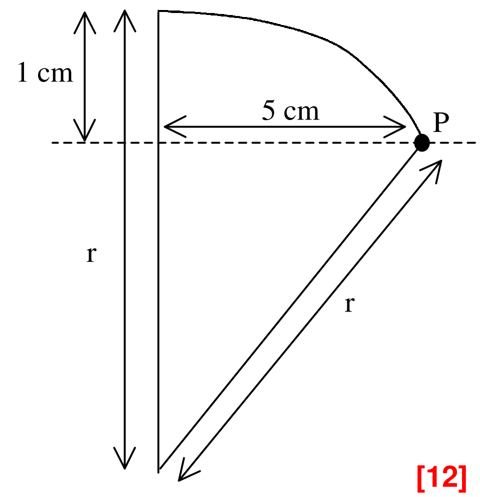
$e = 1.602 \times 10^{-19}$ C

$m_e = 9.11 \times 10^{-31}$ kg $\quad \Rightarrow \quad r = \dfrac{1.71 \times 10^{-4}}{B}$ m.

$v = 3.0 \times 10^{7}$ m s^{-1}

[10]

If the electrons strike the screen at point P, then



$(r - 1)^2 + 5^2 = r^2.$

So $r^2 - 2r + 1 + 25 = r^2.$

$\Rightarrow r = 13$ cm.

Therefore $0.13 = \dfrac{1.71 \times 10^{-4}}{B}$

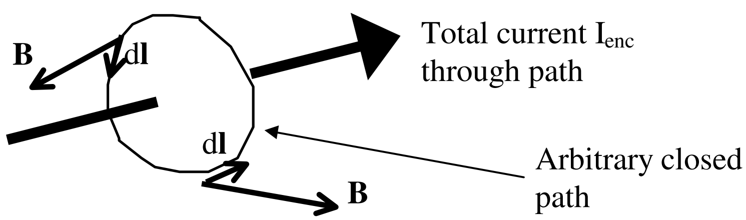
$\Rightarrow \quad B = 1.31$ mT.

[12]

Answer 5

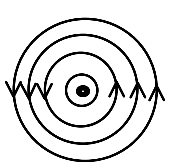
(a) Ampere's Law: $\oint \mathbf{B}.\,\mathbf{dl} = \mu_0 I_{enclosed}$

Ampere's Law states that the line integral of the magnetic field, B, around a closed path is equal to μ_0 times the total current passing through the area bounded by the path:



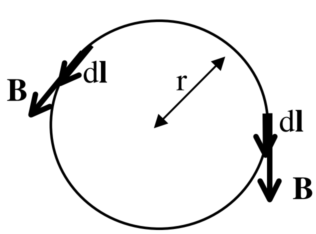
[6]

(b) The magnetic field lines will be circular loops centred on the wire.



View looking along the axis of the wire with I coming out of the page

Consider a circular loop of radius r centred on the wire.



By symmetry, the magnitude of **B** is constant around the loop.

B is everywhere parallel to a line element vector d**l**

$\Rightarrow \mathbf{B.dl} = Bdl$

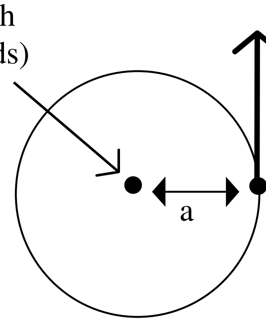
Integrating around the loop, we have

$$\oint \mathbf{B}.\,\mathbf{dl} \;=\; \oint B dl \;=\; B\oint dl \;=\; B(\text{circumference}) \;=\; B(2\pi r) \;=\; \mu_0 I_{enc}$$

The enclosed current is $I_{enc} =$ I since only the current flowing in the wire passes through the loop.

Therefore $B = \dfrac{\mu_0 I}{2\pi r}$. **[8]**

(c) View the two wires along the axial direction.



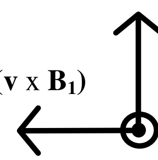
Each wire will be in the magnetic field of the other one.

Say both currents are out of the page.

The magnitude of the field due to wire 1 at wire 2 is $B_1 = \dfrac{\mu_0 I}{2\pi a}$ and its direction perpendicular to the wire as shown.

The current in wire 2 is $I_2 = dQ/dt$ and is constituted by charges moving (out of the page) with speed $v = dl/dt$, where dl is an infinitesimal length of the wire. The force on a charge dQ is

$d\mathbf{F} = dQ(\mathbf{v} \times \mathbf{B_1})$



$\mathbf{v}$ and $\mathbf{B_1}$ are perpendicular so the magnitude of $\mathbf{v} \times \mathbf{B_1}$ is just vB_1.

By the right hand rule, its direction is towards wire 1.

So $dF_{12} = dQvB_1 = (I_2dt)(dl/dt)B_1 = B_1I_2dl$.

For a length of unity, the force is therefore $F_{12} = B_1 I_2 = \dfrac{\mu_0 I_1 I_2}{2\pi a}$. **[10]**

(d) As shown above, the force is attractive if the two currents are in the same direction. If, say, I_2 is in the opposite direction (into the paper) then the direction of $d\mathbf{F} = (\mathbf{v} \times \mathbf{B_1})$ is reversed and points away from wire 1 - i.e., the force is now repulsive.

[6]

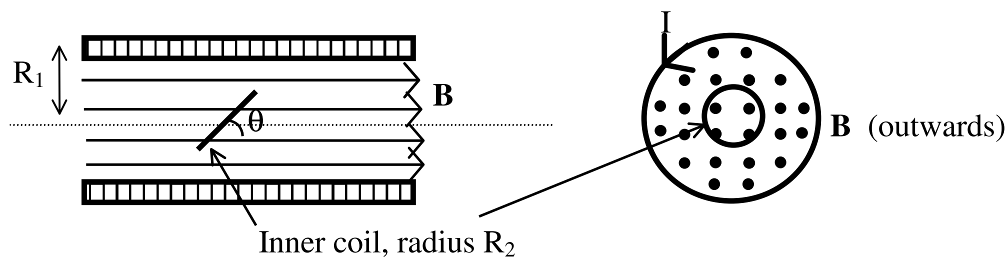
Answer 6

(a) If a circuit carries current I_1 and a second circuit links an amount of magnetic flux Ψ_{21} due to I_1 then the mutual inductance between the circuits is defined as

$M = \Psi_{21}/I_1$ (Inductance = Flux/Current)

The emf induced in the second circuit is $E_2 = -M(dI_1/dt)$. **[6]**

(b) Long solenoid of radius R_1, n turns per unit length, current I_1, $B = \mu_0 n I_1$



SIDE VIEW **AXIAL VIEW**

(i) $$M = \frac{\Psi_{21}}{I_1} = \frac{B(\text{Projected area of inner coil})}{I_1} = \frac{(\mu_0 n I_1)(\pi R_2^2 \sin\theta)}{I_1}$$

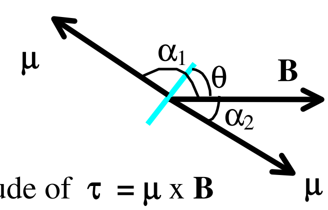
Therefore $M = \mu_0 n \pi R_2^2 \sin\theta$ **[10]**

(ii) Current in inner coil = I_2.

$\Rightarrow$ Magnetic dipole moment vector of inner coil has

Magnitude: $\mu = \pi R_2^2 I_2$ (current)(area)(no. of turns)

Direction: perpendicular to the coil, the exact direction depending on whether the current is clockwise or anticlockwise.

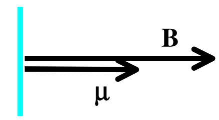


Magnitude of torque on inner coil = magnitude of $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$

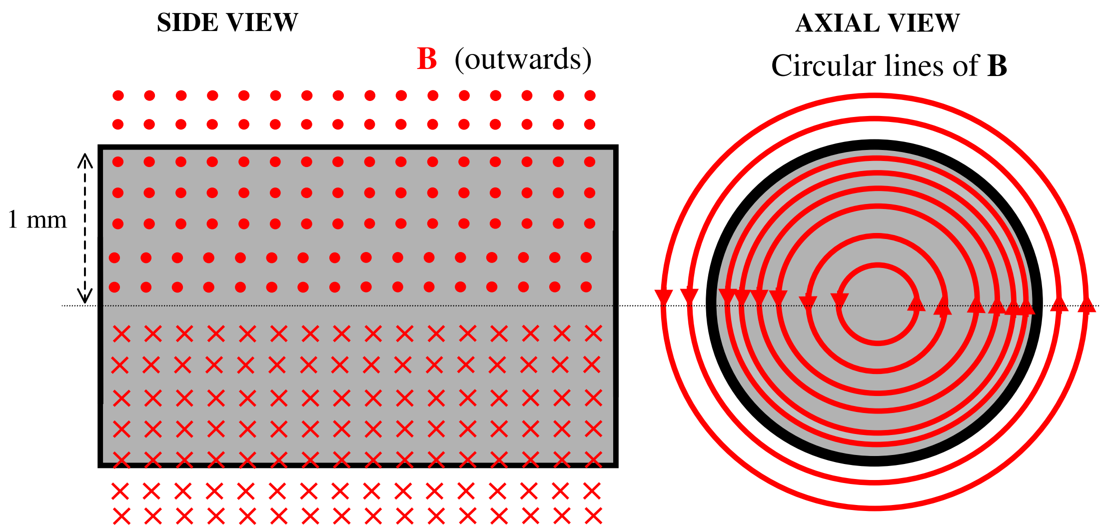
$\tau = \mu B \sin\alpha_1$ or $\mu B \sin\alpha_2 \Rightarrow \tau = \mu B \cos\theta = (\pi R_2^2 I_2)(\mu_0 n I_1)\cos\theta$.

So $\tau = \mu_0 n \pi R_2^2 I_1 I_2 \cos\theta$. **[10]**

(iii) The torque will tend to align the **B** and τ vectors. The coil will therefore come to rest with its plane perpendicular to the magnetic field (i.e., perpendicular to the axis of the solenoid) in which position the torque will be zero.



[4]


SIDE VIEW
B (outwards)
1 mm
AXIAL VIEW
Circular lines of B

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